CONSIDERATION OF HYPERBOLIC HEAT CONDUCTION EQUATION IN RELATION TO LASER SHORT-PULSE HEATING FOR VARIOUS VOLUMETRIC HEAT SOURCES AND BOUNDARY CONDITIONS

BY

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إهداء

أهدي هذا العمل إلى أبي الذي لم يبخل علي يوماً بشيء

وإلى أمي التي غمرتني بالحنان والمحبة ..

أقول لهم: أنتم وهبتموني الطموح والأمل والنشأة على شغف الاطلاع والمعرفة

وإلى إخوتي وأسرتي جميعاً ..

ثم إلى كل من علمني حرفاً أصبح سنا برقه يضيء الطريق أمامي

DEDICATION

This thesis work is dedicated to my dear parents for their continuous support, endless love and encouragement

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“In the name of Allah, Most Compassionate, Ever- Merciful”

Praise is to Allah, Lord of the Worlds. To Him we belong and to Him shall we return. May He send countless blessings and peace upon the noblest of His creatures, His chief emissary, our leader, Prophet Muhammad ﷺ, and upon all of his brethren of prophets and messengers, his family and his companions.

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ABSTRACT

Full Name : Hassan Rashed Al-Duhaim

Thesis Title : Consideration of hyperbolic heat conduction equation in relation to laser short-pulse heating for various volumetric heat sources and boundary conditions

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The heat conduction in solid materials using the hyperbolic heat conduction equation has been studied incorporating laser volumetric sources of two types. In the first case, the laser heating source has been modelled as a time exponentially varying heat pulse while in the second case a step input laser pulse has been used. The thermal stress analysis has been carried out in each of the above cases. The Laplace transform in time and the Fourier cosine transform in space variable have been used to solve these problems. We also studied a mixed boundary value problem arising from heating of a half space with two parts of the boundary satisfying two different conditions. This problem has been solved using the Jones modification of the Wiener-Hopf technique.

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ملخص الرسالة

الاسم الكامل : حسن بن راشد آل دحيم

عنوان الرسالة: الحرق الحراري عن طريق مصادر ليزرية و شروط حدية مختلفة باستخدام النموذج الزائدي لمعادلة انتقال الحرارة.

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في هذه الرسالة تم اعتبار النموذج الزائدي لمعادلة انتقال الحرارة ، وذلك لحرق مادة متناهية الصغر (Nano) باستخدام مصدرين ليزريين مختلفين. في الحالة الأولى تم اختيار نموذج لمصدر من نوع أسي بينما في الحالة الثانية تم اختيار مصدر (المدخل الدرجي). في كلا الحالتين تمت دراسة وتحليل الإجهاد الحراري الناتج عن حرق المادة ، وكمنهجية لهذا العمل تم استخدام محولات لابلاس وفوريي لحل هذه المعادلات. في الفصل الأخير من هذه الرسالة ، تم مناقشة معادلة تفاضلية جزئية ذات شروط حدية مختلطة في نصف مستوي حيث تم حلها باستخدام (تعديل جونس لتقنية فينر و هوبف).

CHAPTER 1

INTRODUCTION TO HEAT CONDUCTION IN SOLIDS

1.1 General introduction

The problem of heat conduction in solids is important due to many engineering applications. One can find formulation and solution of many such problems in for example, Carslaw and Jaeger [3] or Ozisik [14]. In these studies, however, the Fourier law of heat conduction is used to obtain the parabolic heat equation. While this law has proven to be adequate in most situations, the problem involving volumetric sources demand some modifications. The Cattaneo model has proven to be quite suitable in modelling such situations. This approach leads to a hyperbolic heat conduction equation which will be focus of our study. The hyperbolic heat equation can be derived from the Boltzman transport equation using electron kinetic theory approach.

There are many situations where the material is heated suing sources. Of interest are laser heating sources applied on the surface of the material. We have attempted to model such sources as time exponentially varying laser pulse and step input volumetric source in the hyperbolic model in case of two different boundary conditions. Another, feature of heating of solids arises as the change in temperature produces the stresses which are known thermal stresses. In the above cases of laser heating, we have considered the problem of the thermal stress distribution also. This coupling of temperature and thermal stress distribution gives rise to a non-homogenous hyperbolic heat equation and with a wave equation in thermal stress. These problems are tackled using the Laplace in time and the Fourier cosine transform in space variables. The Laplace inversion has been performed analytically while the inverse Fourier cosine transform is obtained using Mathematica.

Laser short pulse treatment of surfaces provides hardness increase at the solid substrate surface and finds wide application in industry due to precision of operation, low cost, and high speed processing. The determination of temperature distribution in solids subjected to a short pulse heating is one of the important problems for crack free hardening of the surfaces due to the attainment of the vast change of temperature and thermal stress fields in the heated region. In the classical theory, heat conduction problem is formulated based upon the Fourier law Carslaw and. Jaeger [3]. In many recent studies, however, it has been shown that the infinite speed of heat transfer predicted by the Fourier law is not valid in problems arising from heat sources with short durations such as laser short pulse irradiation. Since laser intensity is absorbed in the skin of the solid substrate surface, the depth of absorption becomes comparable to the mean free path of phonons. This in turn results in non-equilibrium heating situation in space and time. To formulate such heating problem, several models are presented, which include phase lag and dual phase lag models by Tzou, [15]. In general, heating situation is formulated through the hyperbolic heat conduction model for the consideration of short duration and small spaces. One of the applications of hyperbolic heating situations includes the short pulse high intensity laser irradiation of the solid surfaces. In this case, electrons absorb energy from the irradiated field and transfer some of their excess energy to lattice site through the collisional process Yilbas [21]. Since electron specific heat capacity is less than the lattice specific heat capacity of the substrate material, Kittel [10], thermal separation takes place between electron and lattice sub-systems. Consequently, two equations govern the energy transfer in the solid due to the thermal separation of the electron and the lattice sub-systems during the short heating period. The governing equations are combined through electron-phonon coupling, which allows thermal communication of the both sub-systems. This situation is well presented in the previous study by Yilbas and Al-Aqeeli [18] and the governing heat equation for the lattice sub-system can be reduced form the coupling of both equations.

The laser short-pulse heating of metallic surfaces has generated a lot of interest in the past few decades. Many researches have investigated the phenomenon using different approaches. Hector, Kim and Ozisik [7] have shown that the Fourier law giving rise to the parabolic heat equation is no longer suitable in studying temperature distribution due to a number of engineering situations such as low-temperature conditions or ultrafast heating. They found approximate solution for locked laser pulse using the hyperbolic model. The time dependent form of a unified heat conduction equation has been studied by Lin Et.al. [11] using the method of separation of variables. Yilbas and Pakdemir [17], obtained the approximate solution for the hyperbolic heat conduction equation using the perturbation approach. However, as expected the perturbation solution was valid for a certain range of time and space variables. In another study, Yilas, et al

[20] derived the hyperbolic heat equation from the Boltzmann equation. The authors used the Fourier transform method to find the temperature distribution in closed form in the presence of exponentially decaying laser pulse to deal with practical laser short-pulse heating equation. Duhamel [5] used the finite integral transform method to the hyperbolic heat conduction problem showing that the solution thus obtained provides similar results as in the case of standard transforms. In relativistic settings, Ali and Hany [1] studied the heat conduction by incorporating wave model of heat transfer. They derived the hyperbolic heat equation without any regard to the microstructure of the material. Wang [16] demonstrated the dependence of solution to the temperature distribution hyperbolic model on initial temperature and heat source. Christov [4] formulated the finite speed heat conduction equation using the Maxwell-Cattaneo approach. In [13], Ordonez-Miranda et al studied the thermal relaxation time and the thermal wave oscillations in the hyperbolic model setting. They found the frequency range for strong oscillations in temperature when the thermal relaxation time of finite layer was close to its thermalization time. In [2], Al-Theeb and Yilbas derived the hyperbolic heat equation from the electron kinetic theory and obtained the closed form solution. In another study, Yilbas and Al-Qahtani [19] incorporated the thermal stress fields during the short heating period. They obtained the closed form solution for thermal stress and showed that the stress demonstrated the wave behavior. A closed form solution including heating and cooling cycles with pulse parameter variation was presented by Yilbas and Kalyon [34]. Analytical solution for laser heating was possible using the Laplace transformation method. The closed form solution for hyperbolic heat equation was presented by Yilbas and Pakdemir [33]. They used a perturbation method to solve the heat equation and indicated that temporal behavior of the electron temperature was similar to that corresponding to the laser pulse intensity, provided that both curves had different temporal gradients. Laser evaporative heating and exact solution for temperature rise due to step input pulse was studied by Al-Qahtani and Yilbas [32]. They demonstrated that found temperature rose rapidly in the early heating period because of the energy gain by the substrate material from the irradiated area and only a little energy diffused from the surface region to the solid bulk during the heating pulse. Laser pulse heating of steel surface and flexural wave analysis was investigated by Yilbas et al. [31]. They demonstrated that the dispersion effect of the work piece material, interference of the reflected beam, and partial overlapping of second mode of the travelling wave enabled to identify a unique pattern in the travelling wave in the substrate.

Thermal stress development and entropy change during laser pulse heating of a steel surface was studied by Keles [30]. He showed that the thermal stress field did not follow the temperature distribution inside the substrate material and thermal stress components in the region close to the symmetry axis were compressive. The closed form solution of entropy generation due to laser short pulse heating and consideration of the Seebeck effect was presented by Kassas [29]. He showed that found that a noticeable change in the Seebeck coefficient occurred across the layers despite the smooth changes took place in the electron temperature. Laser repetitive pulse heating and phase change at the irradiated surface was studied by Shuja [28]. He indicated that the size of the melt zone was larger than that of the mushy zone, which was particularly true along the symmetry axis.

Furthermore, the problem of heat conduction also occurs with mixed boundary situation which requires the use of Wiener-Hopf technique. These problems are normally tackled via the use of Jones' modification [9] which is an easy to-use version of the so-called Wiener-Hopf technique. For instance, the analytical solution of a heat conducting problem after considering a mixed boundary value problem arising from layered media that have mixed interfaces using the Jones's method was obtained by Zaman and Al-Khairy [23]. An extension to the work of the paper by Georgiadis et al [6] was done by Zaman [22], where he considered a layered plate consisting of two layers of different materials and uniform thickness via the use of Weiner-Hopf technique. One can find formulation and solution of many such problems in for example, [24], [25], [26] and [27].

1.2 Basic Definitions of Some Terms

Following are some definitions to be used in the thesis;

1.2.1 Integral transforms

We define the following integral transforms as to be used in the work;

1.2.1.1 Laplace transform

The Laplace transform is to be taken in the time variable t of the temperature and thermal stress functions, $U(s,t)$ being defined in$(\,0,\infty)$.

It is defined by,

$$L\{U(s,t)\} = U^*(s,p) = \int_0^\infty U(s,t)e^{-pt}dt.$$

Similarly, the inverse Laplace transform in the transformed Laplace parameter p is to be taken while inverting the transformed function in terms of t defined by,

$$L^{-1}\{U^*(s,p)\} = U(s,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} U^*(s,p)e^{pt}dp.$$

1.2.1.2 Fourier transforms

A Fourier Cosine transform

The Fourier cosine transform is to be taken in the space variable x of the temperature and thermal stress functions $u(x,t)$ in chapter 2 and 3 being defined from(0 to ∞).

It is defined by,

$$F_c\{u(x,t)\} = U(s,t) = \int_0^\infty u(x,t)\cos(sx)\,dx$$

and the inverse Fourier cosine transform by,

$$F_c^{-1}\{U(s,t)\} = u(x,t) = \frac{2}{\pi}\int_0^\infty U(s,t)\cos(sx)\,ds.$$

B Fourier transform

The Fourier transform of a function $u(x,t)$ also is to be taken in the space variable x in chapter 4 in the interval $(-\infty,\infty)$ and is defined by,

$$F\{u(x,t)\} = U(s,t) = \int_{-\infty}^\infty u(x,t)e^{i\alpha x}dx.$$

The inverse Fourier transform is given by,

$$F^{-1}\{U(s,t)\} = u(x,t) = \frac{1}{2\pi}\int_{-\infty}^\infty U(s,t)e^{-isx}ds.$$

1.2.2 Boundary Value Problems

To obtain a temperature distribution and the corresponding thermal stress, it is necessary to solve the governing heat conduction equation and the thermal stress equation. However, to find a complete solution of the equations under consideration, the initial and the boundary conditions must be given. Boundary conditions are mathematical equations describing what takes place physically on the boundary, while initial conditions describe the temperature distribution at time t = 0. Such combination of the equation and the boundary conditions together with the initial conditions is what we refer to boundary value problem.

1.2.3 Jones's Modification of Wiener – Hopf technique

The typical problem that may be solved using Weiner – Hopf technique involves the solution of equations which only give explicit information in a semi-infinite range. This technique was introduced, as a means to solve an integral equation of certain form, by Norbert Wiener (1894-1964) and Eberhard Hopf (1902-1983). However, in a mixed boundary value problem, that this research will be considered, we may know the boundary value of one combination of the unknown functions for $x \geq 0$, and of different combination for $x \leq 0$.

In 1952, D.S. Jones modified Wiener – Hopf technique to solve a mixed boundary value problem directly without transforming those boundaries to equivalent integral equations. The procedure of Jones's method described in the book by B. Noble [12], "Methods Based on The Weiner – Hopf technique" will be considered in this research.

1.2.3.1 Wiener – Hopf Decomposition by contour integration

The main difficulty in using Wiener – Hopf technique is the issue of constructing a suitable factorization. Two major results in decomposition and factorization will be considered and they are all based on Cauchy integrals.

The typical problem obtained by applying some integral transform to partial differential equations is the following, B. Noble [12], pages (36-37):

Find the unknown functions $\emptyset_{+}(\alpha)\ and\ \varphi_{-}(\alpha)$ satisfying:

$$A(\alpha)\emptyset_+(\alpha) + B(\alpha)\varphi_-(\alpha) + C(\alpha) = 0, \tag{1.1}$$

Where ($\alpha = \sigma + i\tau$), is an integral transform parameter.

Equation (1.1) holds in a strip $\tau_- < \tau < \tau_+$, of the complex $\alpha - Plane$.

The function $\emptyset_+(\alpha)$ is analytic in the upper half plane $\tau_- < \tau$, while $\varphi_-(\alpha)$ is analytic in the lower half plane $\tau < \tau_+$. The functions $A(\alpha), B(\alpha), C(\alpha)$ are given functions of α analytic in the strip.

a. Decomposition theorem B. Noble [12], theorem-B, page (13):
   Based upon the hypothesis of theorem B, $f(\alpha)$ can be decomposed as:

$$f(\alpha) = f_+(\alpha) + f_-(\alpha), \tag{1.2}$$

where,

$$f_+(\alpha) = \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic} \frac{f(z)}{z-\alpha} dz, \tag{1.3}$$

and,

$$f_-(\alpha) = \frac{-1}{2\pi i}\int_{-\infty+id}^{\infty+id} \frac{f(z)}{z-\alpha} dz, \tag{1.4}$$

where $f_+(\alpha)$ is analytic for all $\tau_- < \tau$, and $f_-(\alpha)$ is analytic for all $\tau < \tau_+$.

b. Factorization theorem B. Noble [12], theorem-C, page (15):
   Under the hypothesis of theorem C, $K(\alpha)$ can be factorized as:

$$K(\alpha) = K_+(\alpha)K_-(\alpha), \tag{1.5}$$

where $K_+(\alpha)\ and\ K_-(\alpha)$ are analytic, bounded and nonzero in $\tau_- < \tau, \tau < \tau_+$, respectively.

In chapter 2, the closed form solution for temperature distribution due to the time exponential varying laser pulse is obtained. Development of thermal stress is also studied in the irradiated region incorporating the coupled hyperbolic heat and the thermal stress equations. In the second part of this chapter, a convective boundary condition is considered under the same laser volumetric source. In these cases, a double transform consisting of the Laplace transform in time and the Fourier cosine transform in space variable are utilized for the analytical solutions of heat and stress equations. In chapter 3, the closed form solutions of the governing equations of temperature and thermal stress are obtained due to a step input laser pulse. The inverse transforms are computed analytically and Mathematica has been used to get graphical results. In chapter 4, we consider a two dimensional parabolic heat conduction equation incorporated with two types of heat sources. In this context, we study the thermal stress that arises in a mixed boundary setting using the Jones' modifications of the so-called Wiener-Hopf technique. In this work, the hyperbolic heat conduction equation derived from the electron kinetic theory approach [20] is considered and the volumetric heat source due to a laser pulse is incorporated.

Chapter 2

Determination of Temperature Distribution and Thermal Stress with Two Different Boundary Conditions due to Time Exponentially Varying Volumetric Source.

2.1 Introduction

In the first part of this chapter, we consider the hyperbolic heat conduction model and obtain the analytical solution for the laser short-pulse heating of a solid surface. In order to account for the absorption of the incident laser energy, a volumetric source is incorporated in the analysis. Also, it is assumed that there is no convection in the boundary considered.

In the second part we consider convective boundary condition and obtain the closed form solution of temperature and thermal stress distributions under the same laser heat source. The Laplace transform in time and the Fourier cosine transform in space variable are employed to find solution of the problem in the transformation domain. The inversion of the solution from the transform plane is carried out using an analytical approach. We also consider thermal stress development in the irradiated region due to the presence of the volumetric heat source. It is found that temperature rise at the surface follows almost the laser pulse behavior and decay of temperature is sharp in the region next to the surface vicinity of the substrate material. Thermal stress is compressive in the surface region and shows wave behavior with progressing time.

2.2 Determination of temperature distribution and thermal stress for hyperbolic heat conduction equation due to Laser Short-Pulse Heating.

2.2.1 Formulation of the Problem

Consider the laser short-pulse heating situation where the volumetric source resembling the laser pulse is incorporated. The governing hyperbolic heat conduction equation can be written as [20]:

$$A\frac{\partial^2}{\partial t^2}T(x,t) + B\frac{\partial}{\partial t}T(x,t) = \frac{\partial^2}{\partial x^2}T(x,t) + I_0\delta e^{-\delta x}\left(e^{-\beta t} - e^{-\gamma t}\right), \tag{2.1}$$

where $A = \frac{\rho C_p}{k\tau}$ and $B = \frac{\rho C_p}{k}$.

The initial and boundary conditions are given by:

$$T(x,0) = 0,$$

$$\frac{\partial}{\partial t}T(x,0) = 0, \tag{2.2}$$

In Eq. (2.2), it is assumed initially that temperature is zero inside the material and its time derivative also vanishes.

$$\lim_{x\to\infty} T(x,t) = 0,$$

$$\frac{\partial}{\partial x}T(0,t) = 0. \tag{2.3}$$

In Eq. (2.3) at large depth below the surface of the material it is assumed that temperature remains as the initial. Also, there is no convection boundary is assumed at the surface during the short heating period.

Let $D = \left(A\frac{\partial^2}{\partial t^2} + B\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)$ then Eq. (2.1) becomes

$$DT = I_0\delta e^{-\delta x}\left(e^{-\beta t} - e^{-\gamma t}\right). \tag{2.4}$$

Eq. (2.4) can be split into two equations:

$$Du = I_0 \delta e^{-\delta x} e^{-\beta t}, \tag{2.5}$$

$$Dv = I_0 \delta e^{-\delta x} e^{-\gamma t}, \tag{2.6}$$

where, u and v satisfy the same initial and boundary conditions Eq. (2.2) and Eq. (2.3).

Since D is a linear operator, the super-position principle can be applied to get the solution of Eq. (2.4)

i.e. $D(u - v) = Du - Dv = DT$,

hence, $T(x,t) = u(x,t) - v(x,t)$. (2.7)

2.2.2 Solution of the Problem

Applying the Fourier cosine transform in x to Eq. (2.5) and taking into account the vanishing of temperature and its derivative for large x, we get

$$A \frac{d^2}{dt^2} U(s,t) + B \frac{d}{dt} U(s,t) = -s^2 U(s,t) - u_x(0,t) + \frac{I_0 \delta e^{-\beta t}}{s^2+\delta^2}. \tag{2.8}$$

Applying the transformed boundary conditions for Eq. (2.8) we obtain

$$A \frac{d^2}{dt^2} U(s,t) + B \frac{d}{dt} U(s,t) = -s^2 U(s,t) + \frac{I_0 \delta e^{-\beta t}}{s^2+\delta^2}. \tag{2.9}$$

This equation can be solved by applying the Laplace transform with respect to t, so that,

$$Ap^2 U^*(s,p) + BpU^*(s,p) - AU_t(s,0),$$

$$-U(s,0)(Ap + B) = -s^2 U^*(s,p) + \frac{I_0 \delta}{(p+\beta)(s^2+\delta^2)}. \tag{2.10}$$

Now, applying the transformed initial conditions to Eq. (2.10) we get,

$$Ap^2 U^*(s,p) + BpU^*(s,p) = -s^2 U^*(s,p) + \frac{I_0 \delta}{(p+\beta)(s^2+\delta^2)}. \tag{2.11}$$

Hence, $U^*(s,p)$ is given by,

$$U^*(s,p) = \frac{I_0\delta}{(p+\beta)(s^2+\delta^2)(Ap^2+Bp+s^2)}. \tag{2.12}$$

Eq. (2.12) is in the Laplace and Fourier cosine domain and to get this solution in time and space domain we need to take the inverse Laplace and inverse Fourier cosine transforms.

Eq. (2.12) can be written as

$$U^*(s,p) = \frac{I_0\delta}{(s^2+\delta^2)}\left(\frac{a}{P+\beta} + \frac{b}{P-r_1} + \frac{c}{P-r_2}\right), \tag{2.13}$$

where

$$r_1 = \frac{-B+\sqrt{B^2-4As^2}}{2A} \text{ and } r_2 = \frac{-B-\sqrt{B^2-4As^2}}{2A} \quad .$$

The constants a, b and c are due to the partial fraction decomposition, which can found as:

$$a = \frac{1}{A\beta^2-B\beta+s^2}, \; b = \frac{1}{A\beta^2-B\beta+s^2}\frac{2A\beta-B+\sqrt{B^2-4As^2}}{2\sqrt{B^2-4As^2}} \text{ and } c = -\frac{1}{A\beta^2-B\beta+s^2}\frac{2A\beta-B-\sqrt{B^2-4As^2}}{2\sqrt{B^2-4As^2}}.$$

Hence, the Laplace inversion of Eq. (2.13) is

$$U(s,t) = \frac{I_0\delta}{(s^2+\delta^2)}\left(ae^{-\beta t} + be^{r_1 t} + ce^{r_2 t} \right)$$

$$= \frac{I_0\delta}{(s^2+\delta^2)(A\beta^2-B\beta+s^2)}\left[(e^{-\beta t} - e^{-\left(\frac{B}{2A}\right)t}\cosh\left(\frac{t}{2A}q\right) + \frac{2A\beta-B}{2q}e^{-\left(\frac{B}{2A}\right)t}\sinh\left(\frac{t}{2A}q\right)\right], \tag{2.14}$$

where $= \sqrt{B^2-4As^2}$.

Now, we get $u(x,t)$ by taking the inverse Fourier cosine transform of Eq. (2.14),

$$u(x,t) = \frac{2}{\pi}\int_0^\infty U(s,t)\cos(sx)\,ds. \tag{2.15}$$

The function $U(s,t)$ has two positive singularities at

$$s_1 = \sqrt{B\beta - A\beta^2},$$

provided that $\beta \leq \frac{B}{A}$ and $s_2 = \frac{B}{2\sqrt{A}}$.

However, an investigation of Eq. (2.14), see appendix A3, reveals that s_1 is a removable singularity and so the interval of the integration in Eq. (2.15) will not be affected. (i.e. the range of the integration need not to be split in order to avoid this singularity). Also, a graphical representation of the integrand shows that the integrand oscillates for $s_2 \geq \frac{B}{2\sqrt{A}}$ and thus the contribution to the integral vanishes due to self-cancellation effect for $s_2 \geq \frac{B}{2\sqrt{A}}$ .We can therefore restrict the range of integration and write:

$$u(x,t) = \frac{2}{\pi} \int_0^{\frac{B}{2\sqrt{A}}} U(s,t) \cos(sx)\, ds \ .$$

Similarly, for Eq. (2.6) we have, $Dv = I_0 \delta e^{-\delta x} e^{-\gamma t}$, which can be solved by the same above method to obtain the following:

$$V(s,t) = \frac{I_0 \delta}{(s^2+\delta^2)} \left(a e^{-\gamma t} + b e^{r_1 t} + c e^{r_2 t} \right)$$

$$= \frac{I_0 \delta}{(s^2+\delta^2)(A\gamma^2 - B\gamma + s^2)} \times \left(e^{-\gamma t} - e^{-\left(\frac{B}{2A}\right)t} \cosh\left(\frac{t}{2A} q\right) + \frac{2A\gamma - B}{2q} e^{-\left(\frac{B}{2A}\right)t} \sinh\left(\frac{t}{2A} q\right)\right. \tag{2.16}$$

Where, $q = \sqrt{B^2 - 4As^2}$. so that,

$$v(x,t) = \frac{2}{\pi} \int_0^{\frac{B}{2\sqrt{A}}} V(s,t) \cos(sx)\, ds \tag{2.17}$$

Thus, the final solution of this problem is given by Eq. (2.7) which is

$$T(x,t) = u(x,t) - v(x,t)$$

$$= \frac{2}{\pi} \int_0^{\frac{B}{2\sqrt{A}}} U(s,t) \cos(sx)\, ds - \frac{2}{\pi} \int_0^{\frac{B}{2\sqrt{A}}} V(s,t) \cos(sx)\, ds \tag{2.18}$$

where $U(s,t)$ and $V(s,t)$ are given by Eq. (2.14) and Eq. (2.16) respectively.

These integrals have been evaluated numerically using Mathematica to get the value of $T(x,t)$ at any given (x,t). These results are given from Fig. 2.1 – Fig. 2.6.

2.2.3 Thermal-Stress Study

In this section, we study the thermal stress influence on this heating process due to the full laser pulse volumetric heat source.

The equation of the thermal stress is given by the following PDE []:

$$\frac{\partial^2}{\partial x^2}\sigma(x,t) - \left(\frac{(1+v)(1-2\,v)\,\rho}{E\,(1-v)}\right)\frac{\partial^2}{\partial t^2}\sigma(x,t) = \left(\frac{(1+v)}{(1-v)}\,\rho\,\alpha_T\right)\frac{\partial^2}{\partial t^2}T(x,t), \tag{2.19}$$

Where,

$h = \sqrt{\frac{(1+v)(1-2\,v)\rho}{E\,(1-v)}}$ and

$q = \frac{(1+v)}{(1-v)}\,\rho\,\alpha_T$,

are some physical constants given in the simulation table.

The initial and boundary conditions considered here are the same as in Eq. (2.2) and Eq. (2.3).

That is,

$$\sigma(x,0) = 0,$$

$$\frac{\partial}{\partial t}\sigma(x,0) = 0. \tag{2.20}$$

In Eq. (2.20), initially, it is assumed that thermal stress is zero inside the substrate material and time derivative of stress is also zero.

$$\lim_{x\to\infty}\sigma(x,t) = 0,$$

$$\frac{\partial}{\partial x}\sigma(0,t) = 0. \tag{2.21}$$

Equation (2.21) expresses that the stress at large distance (depth) is zero and that surface stress gradient also vanishes.

On taking the Fourier cosine transform of Eq. (2.19), we get,

$$-s^2\bar{\sigma}(s,t) - \sigma_x(0,t) - h^2 \frac{d^2}{dt^2}\bar{\sigma}(s,t) = q\ \frac{d^2}{dt^2}\bar{T}(s,t). \tag{2.22}$$

Applying the transformed boundary conditions for Eq. (2.22) we obtain,

$$-s^2\bar{\sigma}(s,t) - h^2 \frac{d^2}{dt^2}\bar{\sigma}(s,t) = q\ \frac{d^2}{dt^2}\bar{T}(s,t). \tag{2.23}$$

Hence, Eq. (2.23) can be solved by applying the Laplace transform with respect to t.

Define $L\{\bar{\sigma}(s,t)\} = \bar{\sigma}^*(s,p) = \int_0^\infty \bar{\sigma}(s,t)e^{-pt}dt.$

Applying the Laplace transform to Eq. (2.23) yields:

$$-s^2\bar{\sigma}^*(s,p) - h^2p^2\ \bar{\sigma}^*(s,p) - p\ \bar{\sigma}(s,0) - \bar{\sigma}_t(s,0) = q\ p^2\bar{T}^*(s,p) \tag{2.24}$$

Now, applying the transformed initial conditions to Eq. (2.24) we get:

$$-s^2\bar{\sigma}^*(s,p) - h^2p^2\ \bar{\sigma}^*(s,p) = q\ p^2\bar{T}^*(s,p) \tag{2.25}$$

Hence, we obtain:

$$\bar{\sigma}^*(s,p) = \frac{-q\ p^2}{(h^2p^2+s^2)}\ \bar{T}^*(s,p) \tag{2.26}$$

The solution, Eq. (2.26), is in the transformed domain and to get the solution in time and space domain we need to take the inverse Laplace and inverse Fourier cosine transforms.

Taking the Laplace and Fourier cosines transforms of Eq. (2.7) gives:

$$\bar{T}^*(s,p) = U^*(s,p) - V^*(s,p) \tag{2.27}$$

Substitute Eq. (2.27) into Eq. (2.26) we get:

$$\bar{\sigma}^*(s,p) = \frac{-q\ p^2}{(h^2p^2+s^2)}[U^*(s,p) - V^*(s,p)\ ] \tag{2.28}$$

Now, let

$$\overline{\emptyset_1}^*(s,p) = \frac{-q\,p^2}{(h^2p^2+s^2)}\; U^*(s,p) \tag{2.29}$$

and

$$\overline{\emptyset_2}^*(s,p) = \frac{-q\,p^2}{(h^2p^2+s^2)}\; V^*(s,p). \tag{2.30}$$

Where, U^* and V^* are the functions found earlier in Eq. (2.12),

$$U^*(s,p) = \frac{I_0\delta}{(p+\beta)(s^2+\delta^2)(Ap^2+Bp+s^2)} \text{ and } V^*(s,p) = \frac{I_0\delta}{(p+\gamma)(s^2+\delta^2)(Ap^2+Bp+s^2)}.$$

Now, $\overline{\emptyset_1}^*(s,p)$ becomes:

$$\overline{\emptyset_1}^*(s,p) = \frac{-q\,p^2}{(h^2p^2+s^2)}\,\frac{I_0\delta}{(p+\beta)(s^2+\delta^2)(Ap^2+Bp+s^2)}. \tag{2.31}$$

Thermal stress function $\sigma(x,t)$ is taken to be,

$$\sigma(x,t) = \emptyset_1(x,t) - \emptyset_2(x,t) \tag{2.32}$$

Now, applying Laplace inverse transform on $\overline{\emptyset_1}^*(s,p)$, we get:

$$\overline{\emptyset_1}(s,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \overline{\emptyset_1}^*(s,p)\, e^{pt}\, dp \tag{2.33}$$

where, c is a positive real number which is chosen in such a way that all the real parts of the poles of the integrand are smaller than c.

The integral in Eq. (2.33) can be evaluated using the residue theorem.

Letting

$$F(p) = \frac{(-q\ I_0\delta)\ p^2\ e^{pt}}{(h^2p^2+s^2)(p+\beta)(s^2+\delta^2)(Ap^2+Bp+s^2)} = \overline{\emptyset_1}^*(s,p)\ e^{pt}.$$

F, has five simple poles as follows:

$$P_1 = -\beta,\ P_2 = \frac{i\,s}{h},\ P_3 = \frac{-i\,s}{h},\ P_4 = \frac{-B+\sqrt{B^2-4As^2}}{2A} \text{ and } P_5 = \frac{-B-\sqrt{B^2-4As^2}}{2A}.$$

Therefore,

$$\overline{\emptyset_1}(s,t) = \sum_{j=1}^{5} Res\{F(p), P_j\}. \tag{2.34}$$

Evaluation of the residues gives,

$$Res\{F(p), P_1\} = \lim_{P\to-\beta}\ (P-P_1)\frac{(-q\ I_0\delta)\ p^2\ e^{pt}}{(h^2p^2+s^2)(p+\beta)(s^2+\delta^2)(Ap^2+Bp+s^2)}$$

$$= \frac{(-q\ I_0\delta)}{(s^2+\delta^2)}\ \frac{\beta^2\ e^{-\beta t}}{(h^2\beta^2+s^2)(s^2-B\beta+A\beta^2)},$$

$$Res\{F(p), P_2\} = \lim_{P\to\frac{i\,s}{h}}\ (P-P_2)\ \frac{(-q\ I_0\delta)\ p^2\ e^{pt}}{(h^2p^2+s^2)(p+\beta)(s^2+\delta^2)(Ap^2+Bp+s^2)}$$

$$= \frac{(-q\ I_0\delta)}{(s^2+\delta^2)}\ \frac{i\ e^{\frac{i\,s}{h}t}}{2(is+h\beta)(iBh-As+sh^2)},$$

$$Res\{F(p), P_3\} = \lim_{P\to-\frac{i\,s}{h}}\ (P-P_3)\ \frac{(-q\ I_0\delta)\ p^2\ e^{pt}}{(h^2p^2+s^2)(p+\beta)(s^2+\delta^2)(Ap^2+Bp+s^2)}$$

$$= \frac{(-q\ I_0\delta)}{(s^2+\delta^2)}\ \frac{i\ e^{-\frac{i\,s}{h}t}}{2(-is+h\beta)(-iBh-As+sh^2)},$$

$$Res\{F(p), P_4\} = \lim_{P\to\frac{-B+\sqrt{B^2-4As^2}}{2A}}\ (P-P_4)\ \frac{(-q\ I_0\delta)\ p^2\ e^{pt}}{(h^2p^2+s^2)(p+\beta)(s^2+\delta^2)(Ap^2+Bp+s^2)}$$

$$= \frac{(-q\ I_0\delta)}{(s^2+\delta^2)}\ \frac{2A\ e^{\frac{-B+\sqrt{B^2-4As^2}}{2A}t}\left(-B^2+2As^2+B\sqrt{B^2-4As^2}\right)}{\sqrt{B^2-4As^2}\left(-B^2h^2-2A^2s^2+2h^2As^2+Bh^2\sqrt{B^2-4As^2}\right)\left(-B+\sqrt{B^2-4As^2}+2A\beta\right)},$$

$$Res\{F(p),P_5\} = \lim_{P\to\frac{-B-\sqrt{B^2-4As^2}}{2A}} (P-P_5)\frac{(-q\ I_0\delta)\ p^2\ e^{pt}}{(h^2p^2+s^2)(p+\beta)(s^2+\delta^2)(Ap^2+Bp+s^2)}$$

$$=\frac{(-q\ I_0\delta)}{(s^2+\delta^2)}\ \frac{2A\ e^{\frac{-B-\sqrt{B^2-4As^2}}{2A}t}\left(B^2-2As^2+B\sqrt{B^2-4As^2}\right)}{\sqrt{B^2-4As^2}\left(B^2h^2 2A^2s^2-2h^2As^2+Bh^2\sqrt{B^2-4As^2}\right)\left(B+\sqrt{B^2-4As^2}-2A\beta\right)},$$

Then, Eq. (2.33) becomes:

$$\overline{\emptyset_1}(s,t) =$$

$$\frac{(-q\ I_0\delta)}{(s^2+\delta^2)}\left[\begin{array}{c}\frac{\beta^2\ e^{-\beta t}}{(h^2\beta^2+s^2)(s^2-B\beta+A\beta^2)}+\frac{i\ e^{\frac{i\,s}{h}t}}{2(is+h\beta)(iBh-As+sh^2)}+\frac{i\ e^{-\frac{i\,s}{h}t}}{2(-is+h\beta)(-iBh-As+sh^2)}+\\ \frac{2A\ e^{\left(\frac{-B+\sqrt{B^2-4As^2}}{2A}\right)t}\left(-B^2+2As^2+B\sqrt{B^2-4As^2}\right)}{\sqrt{B^2-4As^2}\left(-B^2h^2-2A^2s^2+2h^2As^2+Bh^2\sqrt{B^2-4As^2}\right)\left(-B+\sqrt{B^2-4As^2}+2A\beta\right)}+\\ \frac{2A\ e^{\left(\frac{-B-\sqrt{B^2-4As^2}}{2A}\right)t}\left(B^2-2As^2+B\sqrt{B^2-4As^2}\right)}{\sqrt{B^2-4As^2}\left(B^2h^2 2A^2s^2-2h^2As^2+Bh^2\sqrt{B^2-4As^2}\right)\left(B+\sqrt{B^2-4As^2}-2A\beta\right)}\end{array}\right]. \tag{2.35}$$

Simplifying the above equation, we get,

$$\overline{\emptyset_1}(s,t) =$$

$$\frac{(-q\ I_0\delta)}{(s^2+\delta^2)}\left(\begin{array}{c}\left[\frac{e^{\left(\frac{-B-\sqrt{B^2-4As^2}}{2A}\right)t}\left(\begin{array}{c}-ABs^2-Bh^2s^2-As^2\sqrt{B^2-4As^2}-As^2\sqrt{B^2-4As^2}\ e^{\left(\frac{\sqrt{B^2-4As^2}}{A}\right)t}+h^2s^2\sqrt{B^2-4As^2}+\\ h^2s^2\sqrt{B^2-4As^2}\ e^{\left(\frac{\sqrt{B^2-4As^2}}{A}\right)t}+AB^2\beta-2A^2s^2\beta+2Ah^2s^2\beta+AB\beta\sqrt{B^2-4As^2}+\\ AB\beta\sqrt{B^2-4As^2}\ e^{\left(\frac{\sqrt{B^2-4As^2}}{A}\right)t}+\\ e^{\left(\frac{\sqrt{B^2-4As^2}}{A}\right)t}\left(Bs^2\left(A+h^2\right)-A\beta B^2+2As^2\beta\left(A-h^2\right)\right)\end{array}\right)}{2\sqrt{B^2-4As^2}(B^2h^2+A^2s^2-2Ah^2s^2+h^4s^2)(B\beta-s^2-A\beta^2)}\right]+\\ \frac{\beta^2\ e^{-\beta t}}{(h^2\beta^2+s^2)(s^2-B\beta+A\beta^2)}+\\ \frac{\left(-As^2+h^2\left(\beta B+s^2\right)\right)Cos\left[\frac{s\,t}{h}\right]+hs\left(B+\beta\left(A-h^2\right)\right)Sin\left[\frac{s\,t}{h}\right]}{(B^2h^2+A^2s^2-2Ah^2s^2+h^4s^2)(h^2\beta^2+s^2)}\end{array}\right)$$

.

Similarly, $\overline{\emptyset_2}(s,t)$ can be obtained by replacing β with γ in Eq. (2.35).

Now, we find $\emptyset_1(x,t)$ by taking the inverse Fourier cosine transform of $\overline{\emptyset_1}(s,t)$,

$$\emptyset_1(x,t) = \frac{2}{\pi}\int_0^{\infty} \overline{\emptyset_1}(s,t)\cos(sx)\,ds. \tag{2.36}$$

$\overline{\emptyset_1}(s,t)$ has two positive singularities at,

$sg_1 = \sqrt{B\beta - A\beta^2}$, provided that $\beta \leq \frac{B}{A}$ and $sg_2 = \frac{B}{2\sqrt{A}}$.

Since sg_1 is a removable singularity, see A3, the range of the integration of Eq. (2.36) will not be affected. An argument similar to that in Eq. (2.15), we can reduce the range of integration and write:

$$\emptyset_1(x,t) = \frac{2}{\pi}\int_0^{\frac{B}{2\sqrt{A}}} \overline{\emptyset_1}(s,t)\cos(sx)\,ds. \tag{2.37}$$

Similarly,

$$\emptyset_2(x,t) = \frac{2}{\pi}\int_0^{\frac{B}{2\sqrt{A}}} \overline{\emptyset_2}(s,t)\cos(sx)\,ds. \tag{2.38}$$

So Eq. (2.32), becomes,

$$\sigma(x,t) = \frac{2}{\pi}\int_0^{\frac{B}{2\sqrt{A}}} \overline{\emptyset_1}(s,t)\cos(sx)\,ds - \frac{2}{\pi}\int_0^{\frac{B}{2\sqrt{A}}} \overline{\emptyset_2}(s,t)\cos(sx)\,ds. \tag{2.39}$$

These integrals were evaluated numerically using Mathematica to obtain $\sigma(x,t)$. The results are given from Fig. 2.7 – Fig. 2.12.

2.2.4 Results and discussion

Analytical solution for the hyperbolic heat conduction equation accounting for finite speed of heat conduction with presence of the volumetric heat source is presented. The short pulse laser source is incorporated as a volumetric source in the equation. We have used the Laplace transform in time and the Fourier cosine transform in space to find the solution of the hyperbolic heat conduction equation incorporating the appropriate initial and boundary conditions. The inversion of the Laplace transform is performed analytically. For inversion of the Fourier cosine transform, the software Mathematica is used and solutions are displayed graphically. In addition, thermal stress field is obtained through coupling heat and thermal stress equations and the integral transforms are used to obtain the solutions of the coupled equations. The inversion of the Laplace transform is performed using complex residue theory while Mathematica is used for the inverse Fourier cosine transform. However, inversion from the transformed plane to the physical plane is involved with complexity. This is because of the singularities of integrand in the solutions. In this case, singularities are removed and avoided through reducing the range of integration, which can be justified due to the oscillatory nature of the integral because of self-cancellation effect.

Fig. 2.1 and Fig. 2.3 shows, the temperature distribution inside the irradiated substrate for different heating periods and for two laser pulse parameter ratio of γ/β, 0.5 and 0.33 respectively. Temperature decays gradually in the surface region and as the depth below the surface increases temperature decay becomes sharp. This is associated with the absorption of the laser beam, which decays exponentially from the surface towards the solid bulk in accordance with the Lambert's Beer law [8]. It should be noted that steel is used as the substrate material and the absorption coefficient of the steel is 6.17×10^{7} 1/m, which is one over the absorption depth. Therefore, the absorption depth of the irradiated intensity is in the order of 1.6×10^{-8} m. Consequently, over 60% of the irradiated energy is absorbed within the absorption depth. This, in turn, increases the internal energy gain of the substrate within this region. Since insulated boundary condition is incorporated at the surface due to short heating period, heat transfer takes place from the surface region towards the solid bulk. The heat transfer by conduction is governed by the temperature gradient and the thermal conductivity of the substrate. Therefore, high temperature gradient accelerates the heat conduction at some depth below the surface rather than in the surface region. This causes gradual decay of temperature in the surface vicinity and sharp decay of temperature at some depth below the surface vicinity. As the heating period progresses, temperature attains high values in the surface region and temperature decay becomes sharper at some depth below the surface as compared to that occurring at the early heating period. However, in the surface region, temperature decay is more gradual for the early heating period than that corresponding to the late heating period despite the fact that thermally insulated boundary condition is incorporated for both cases. This indicates that heat transfer rates due to conduction from the surface region towards the solid bulk enhances with progressing time.

Fig. 2.4 and Fig. 2.5 show temporal variation temperature at different depths below the surface for two laser pulse parameter (γ/β =0.5 and 0.4), respectively. Temperature rise is gradual in the early heating period, which is more pronounced at x = 20 nm below the surface. The gradual temperature rise in the early heating period demonstrates the material thermal response to the laser pulse Fig. 2.6. In this case, temperature rise does not follow the temporal behavior of the laser pulse intensity. Moreover, the slow rise of temperature at some depth below the surface is associated with the absorption of the laser beam intensity at this depth. According to the Lambert's Beer law, exponential decay of laser intensity with increasing depth below the surface results in lower absorbed energy at a depth 20 nm below the surface as compared to that corresponding to the surface vicinity (x = 1 nm). Consequently, internal energy gain from the laser irradiated field becomes small in this region. In addition, conduction heat transfer from surface region towards the solid bulk contributes to the internal energy gain in this region while modifying the temporal response of the temperature increase in this region. In the case of small value of laser pulse parameter (γ/β = 0.33), temperature rise becomes faster than that of the high value of the laser pulse parameter (γ/β = 0.5). This is associated with the laser pulse energy, which becomes larger for small values of the laser pulse parameter Fig. 2.6 In addition, temperature attains higher values for the low laser pulse parameter than that corresponding to the high value of the laser pulse parameter.

Fig. 2.8 shows thermal stress distribution inside the substrate material for three heating periods. Thermal stress is negative in the substrate, which demonstrates the compressive nature. Since stress gradient is set to be zero at the surface, stress remains high in the surface region despite the relatively lower temperature gradient in the surface region as compared to some depth below the surface. The zero stress gradient resemble the stress continuity at the surface as if the substrate surface has a coating and the stress gradient becomes zero at the coating substrate surface interface. In addition, zero stress gradient resembles the mechanical constraints at the surface such that surface is not free to thermally expands during the heating period. This mainly occurs during the laser shock processing in which the optical overlay is placed at the surface to increase the stress magnitude in the surface region. As the depth below the surface increases, stress reduces. This behavior is associated with the temperature gradient, which low values with further increase in depth below the surface. Heating period has significant effect in the stress values below the surface. However, the general trend is that increasing heating period enhances the temperature gradient so that thermal stress increases below the surface. In the case of the laser pulse parameter (0.33), reducing laser pulse parameter results and increased laser peak intensity, this in turn results in high temperature rise and sharp temperature decay inside the substrate material as shown in Fig. 2.3 and Fig. 2.6. As a consequence, thermal stress level increases with reduced laser pulse parameter. Temporal variation of thermal stress at different depths below the surface is shown in Fig. 2.10 and Fig. 2.11 Temporal variation of stress field demonstrates the wave behavior and it propagates into the substrate material as the depth below the surface increases. Since the wave speed ($h = \sqrt{\frac{(1+v)(1-2\,v)\rho}{E\,(1-v)}}$) is kept constant in the analysis, stress wave propagates at a constant speed inside the substrate material. Since substrate material is considered to be stress free initially, zero stress occurs at time t = 0. As the time progresses, stress field is developed, which is compressive.

2.2.5 Tables

Table 2.1: The properties and the range of values used in the simulations

$\alpha\ (\mathrm{m^2/s})$	$\beta\ (1/\mathrm{s})$	$\delta\ (1/\mathrm{m})$	$I_0\ (\mathrm{W/m^2})$	$\gamma\ (1/\mathrm{s})$	$A\ (\mathrm{s^2/m^2})$	$B\ (\mathrm{s/m^2})$
$\mathbf{0.227 \times 10^{-4}}$	2×10^{11}	6.16×10^{7}	4×10^{20}	$4, 5$ and 6×10^{11}	2.1×10^{-10}	44052.86

Table 2.2: The properties and the range of values used in the simulations

$\alpha\ (\mathrm{m^2/s})$	v	$E\ (1/\mathrm{K})$	$\rho\ (\mathrm{kg/m^3})$	$\alpha_T\ (1/\mathrm{s})$	$A\ (\mathrm{s^2/m^2})$	$B\ (\mathrm{s/m^2})$
$\mathbf{0.227 \times 10^{-4}}$	0.3	200×10^{9}	7850	12×10^{-6}	2.1×10^{-10}	44052.86

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2.2.6 Figures

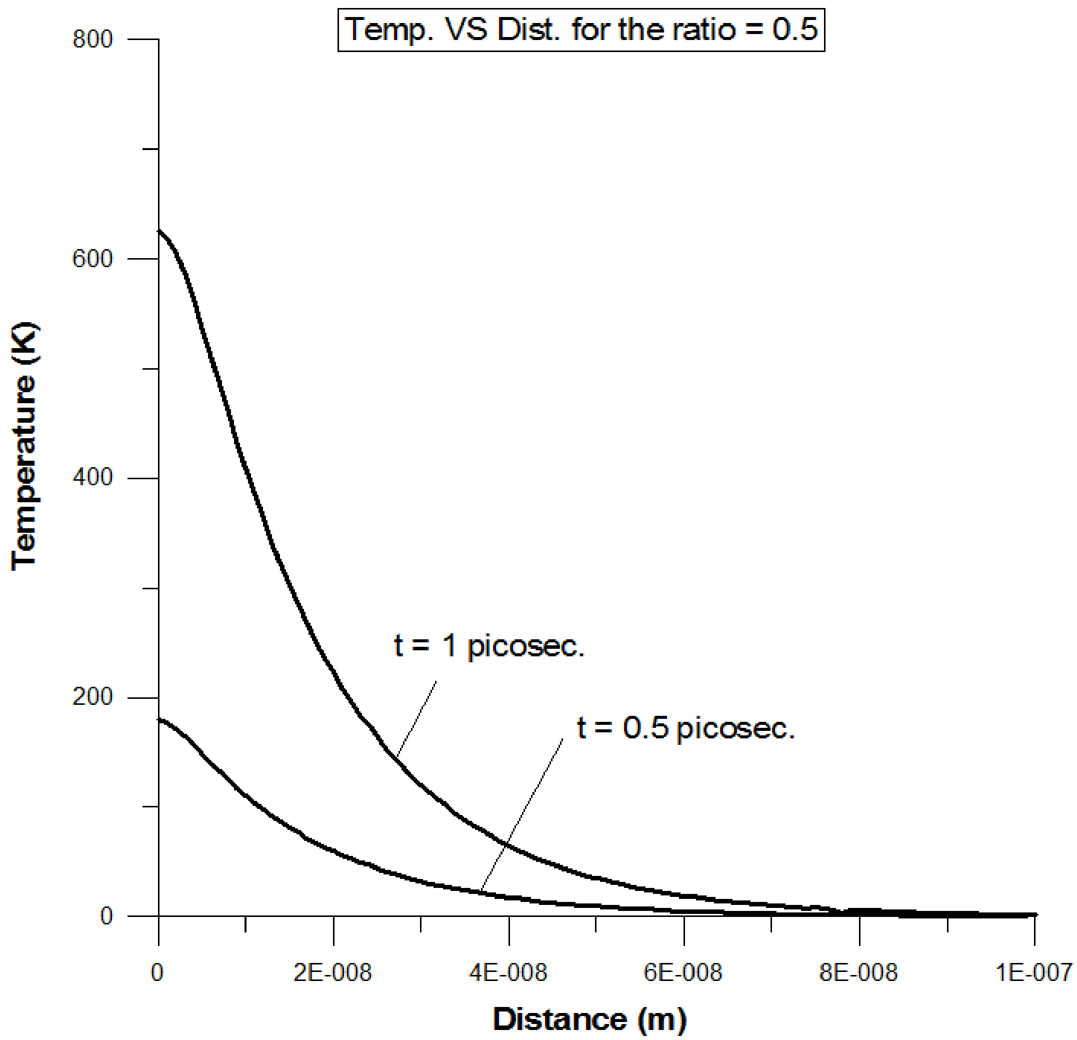

Figure 2.1: Temperature Variation inside the substrate material for different heating periods for the pulse ratio = 0.5

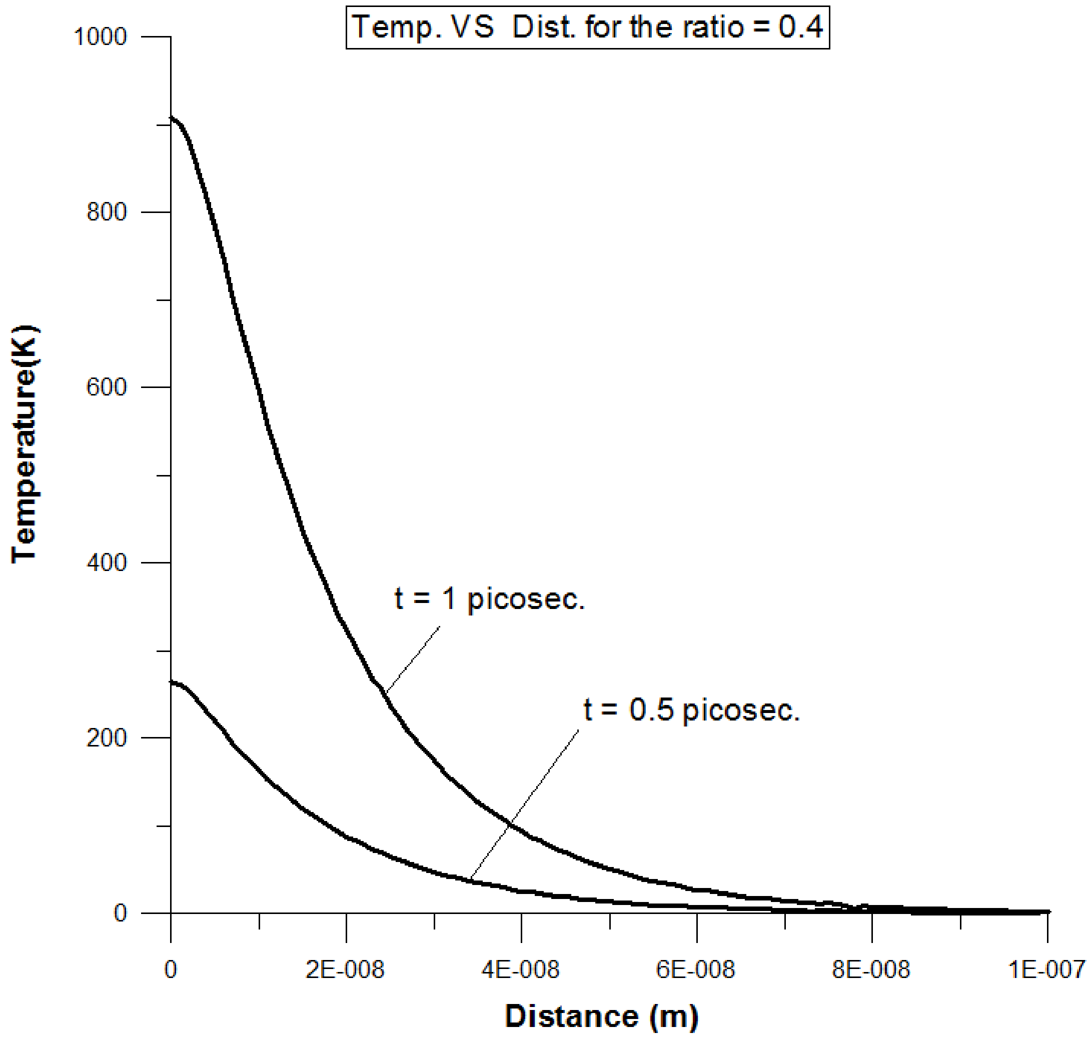

Figure 2.2: Temperature Variation inside the substrate material for different heating periods for the pulse ratio = 0.4

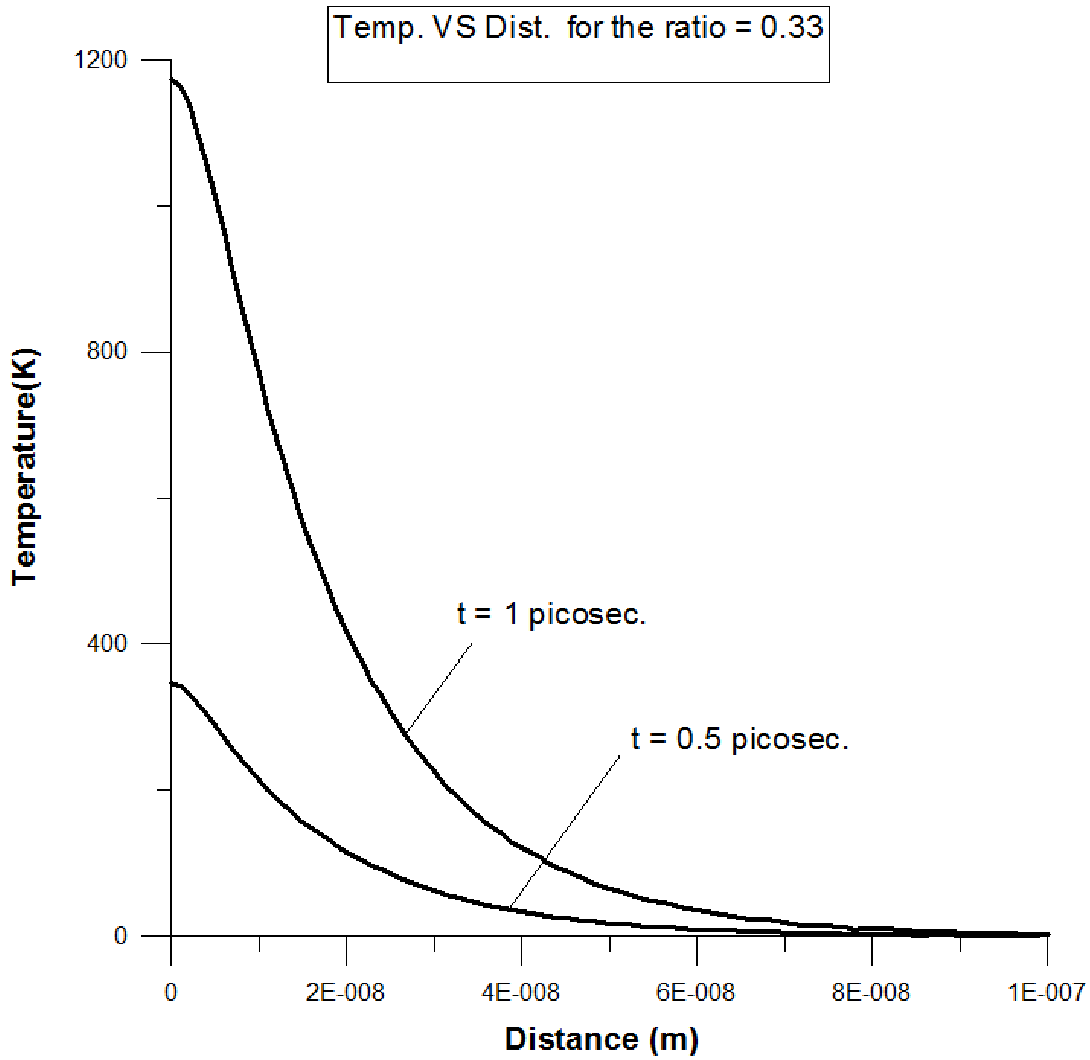

Figure 2.3: Temperature Variation inside the substrate material for different heating periods for the pulse ratio = 0.33

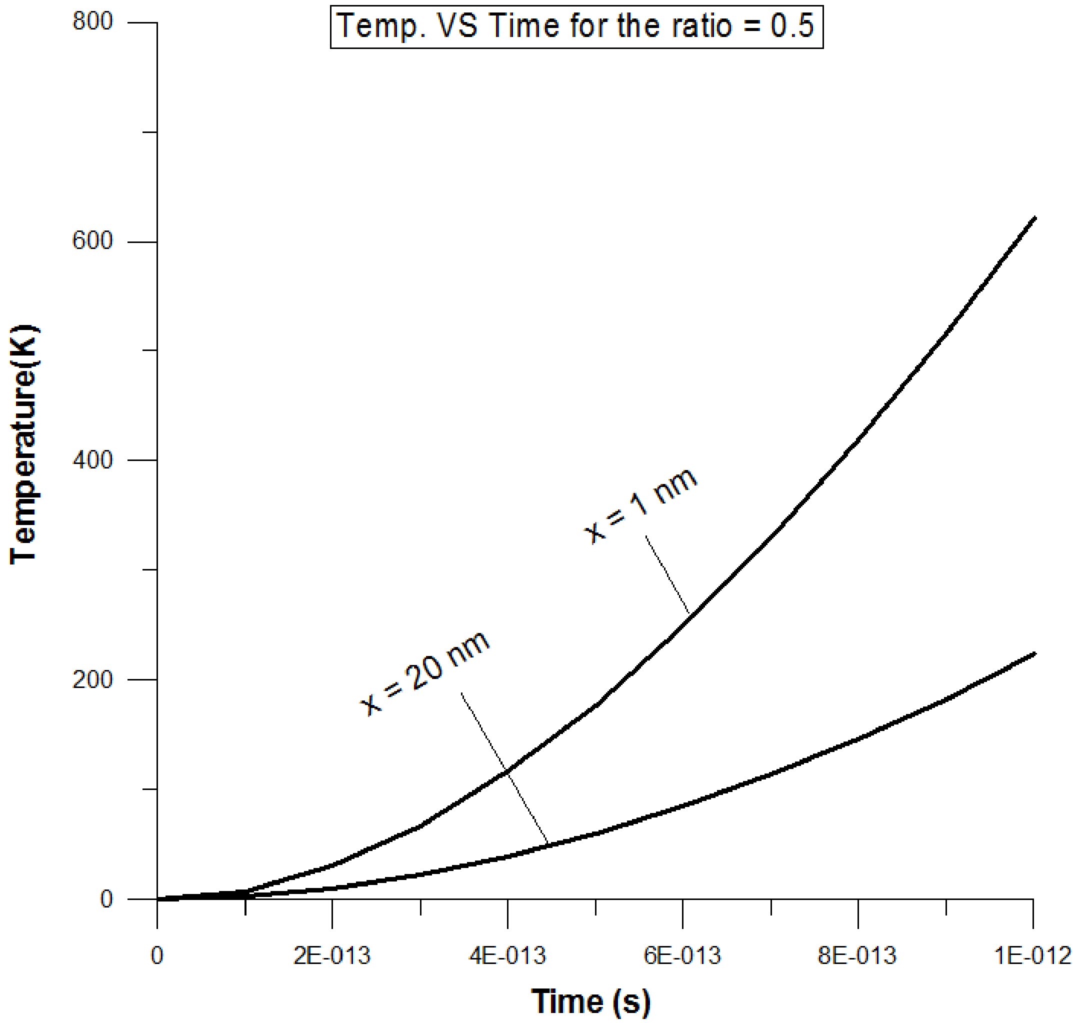

Figure 2.4: Temperature Variation with time at different depths for the pulse ratio = 0.5

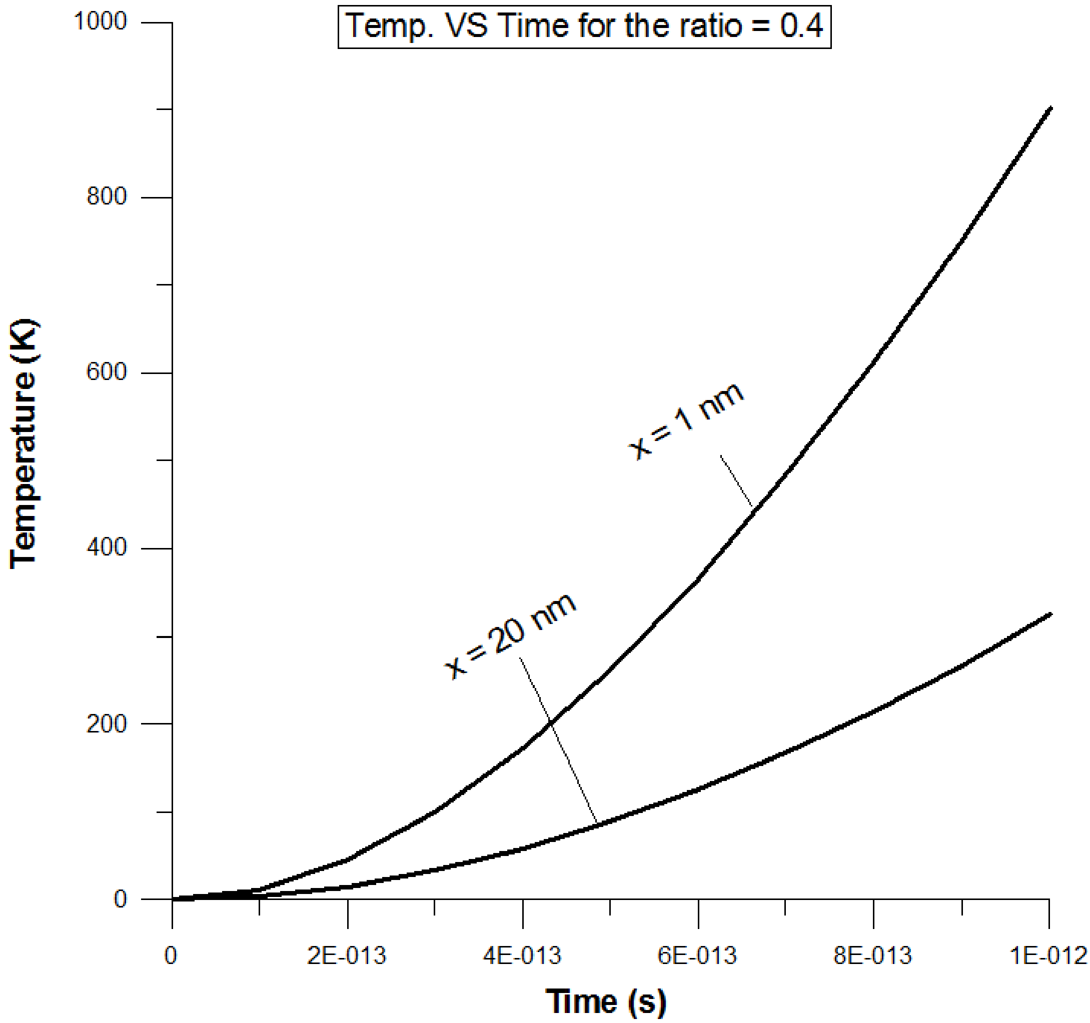

Figure 2.5: Temperature Variation with time at different depths for the pulse ratio = 0.4

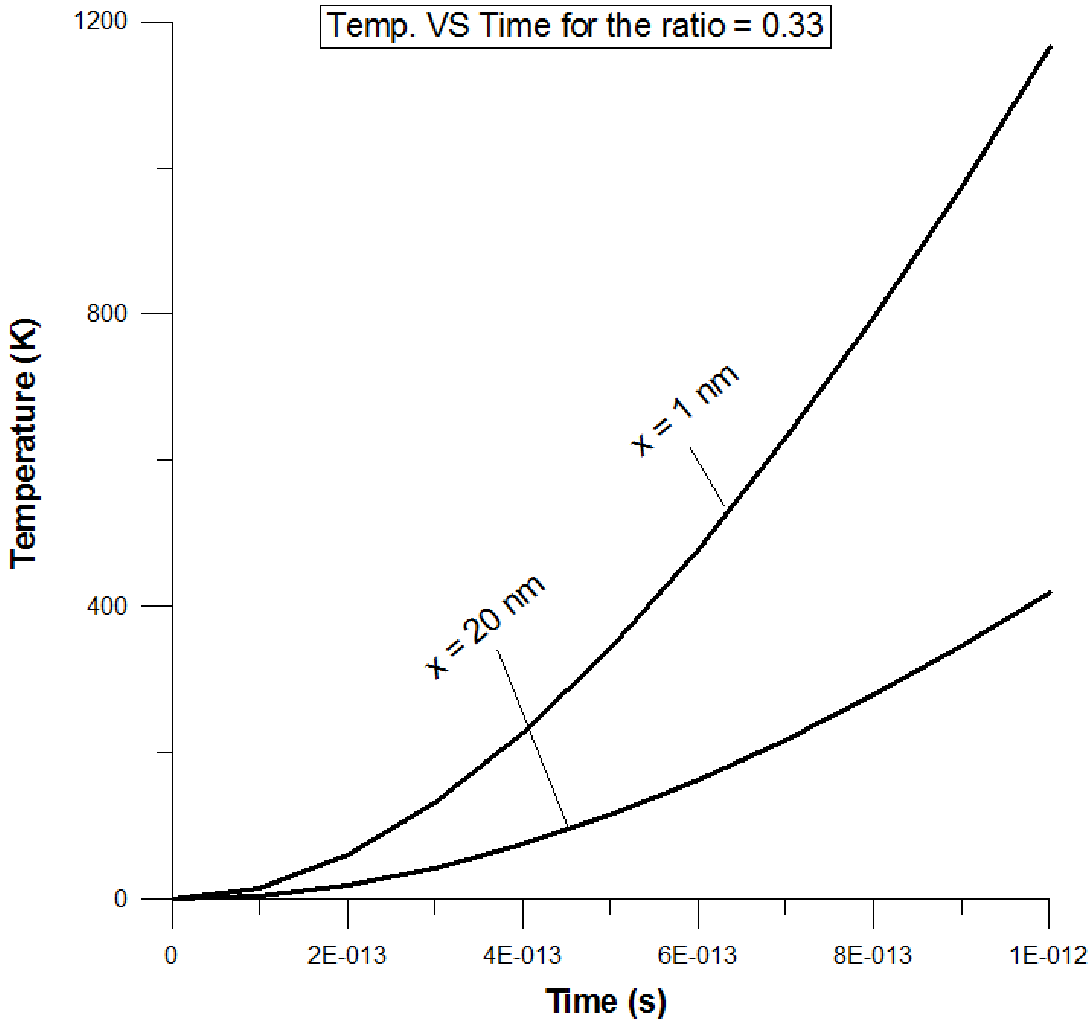

Figure 2.6: Temperature Variation with time at different depths for the pulse ratio = 0.33

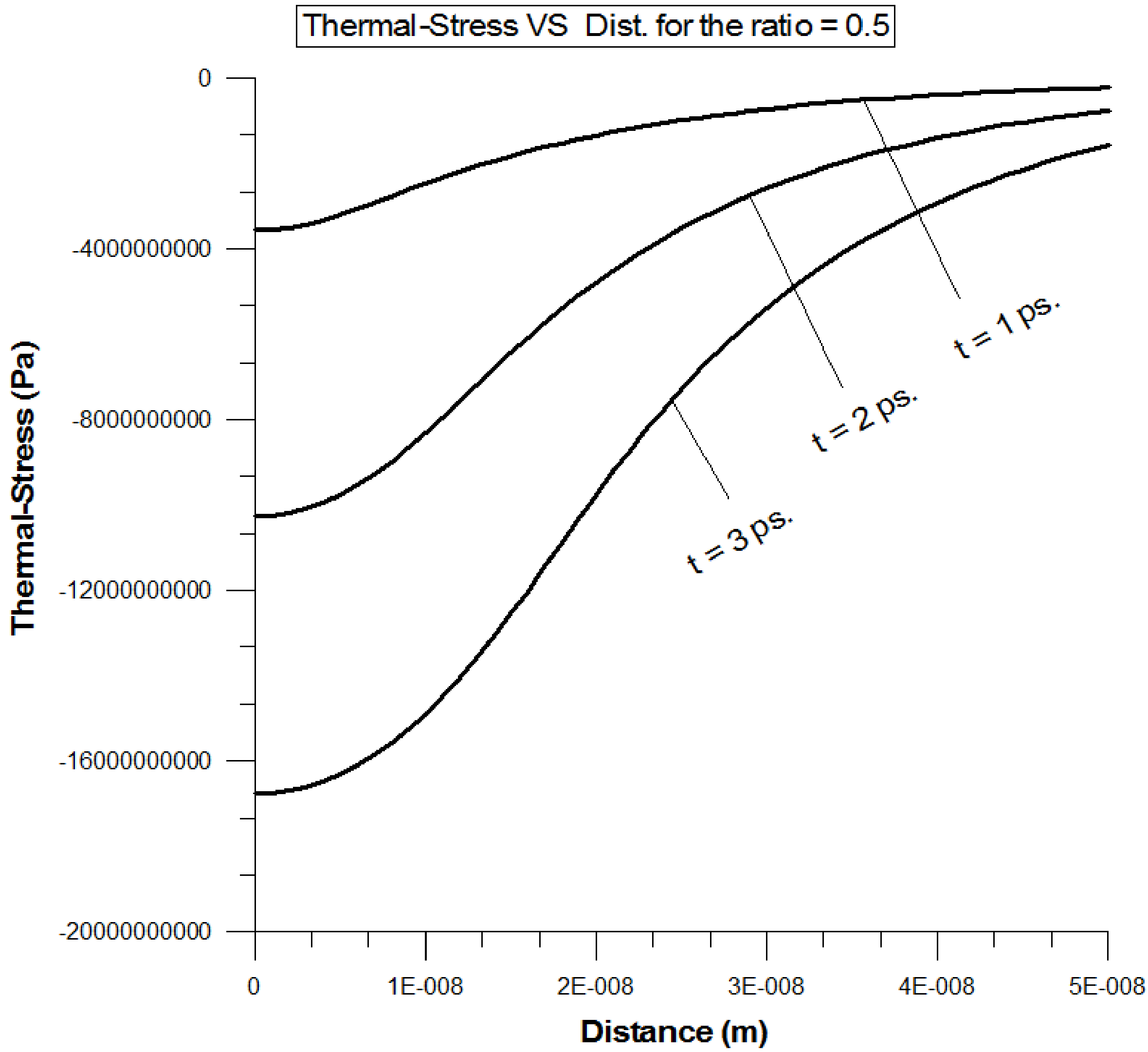

Figure 2.7: Thermal-Stress Variation inside the substrate material for different heating periods for the pulse ratio = 0.5

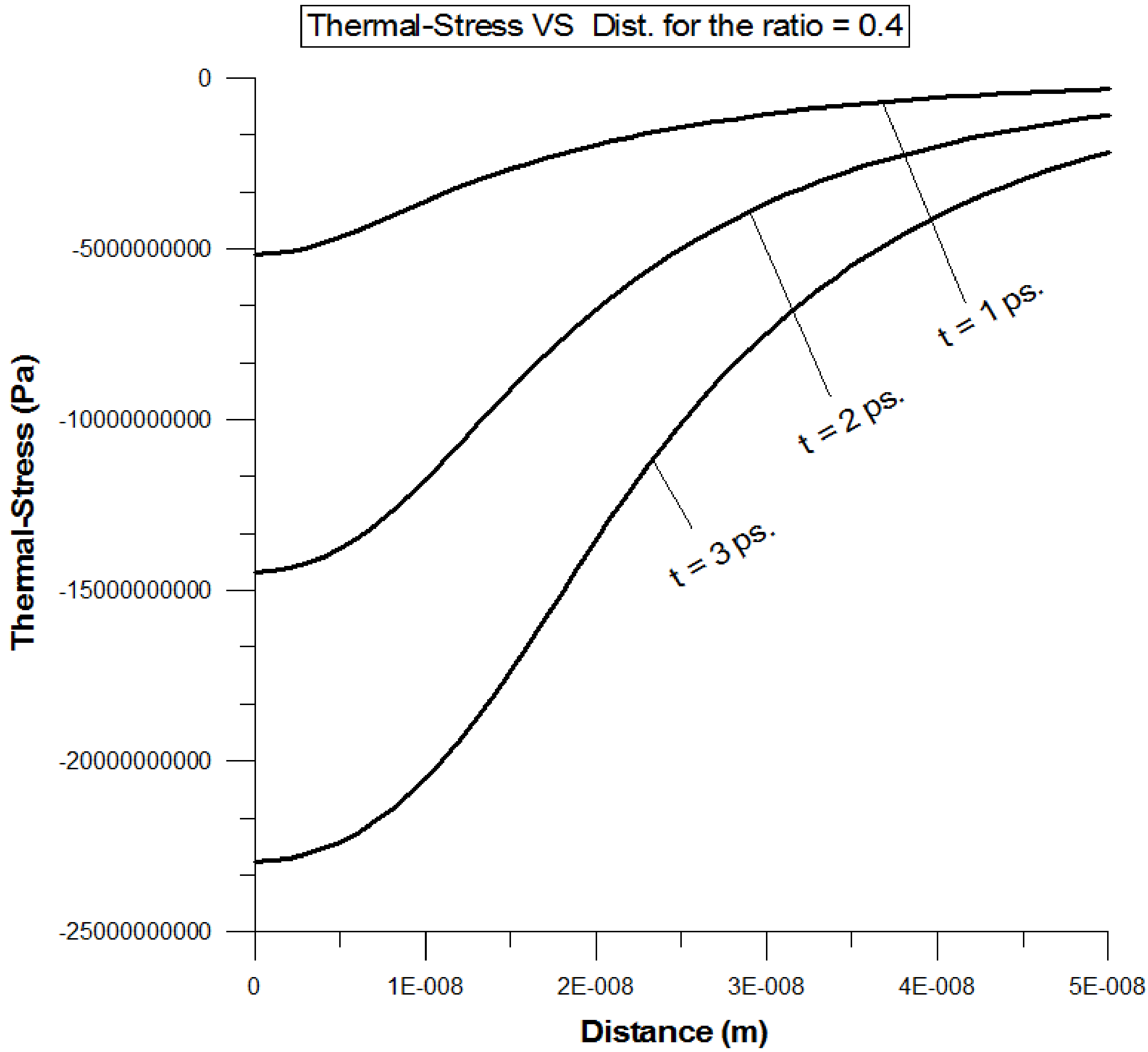

Figure 2.8: Thermal-Stress Variation inside the substrate material for different heating periods for the pulse ratio = 0.4

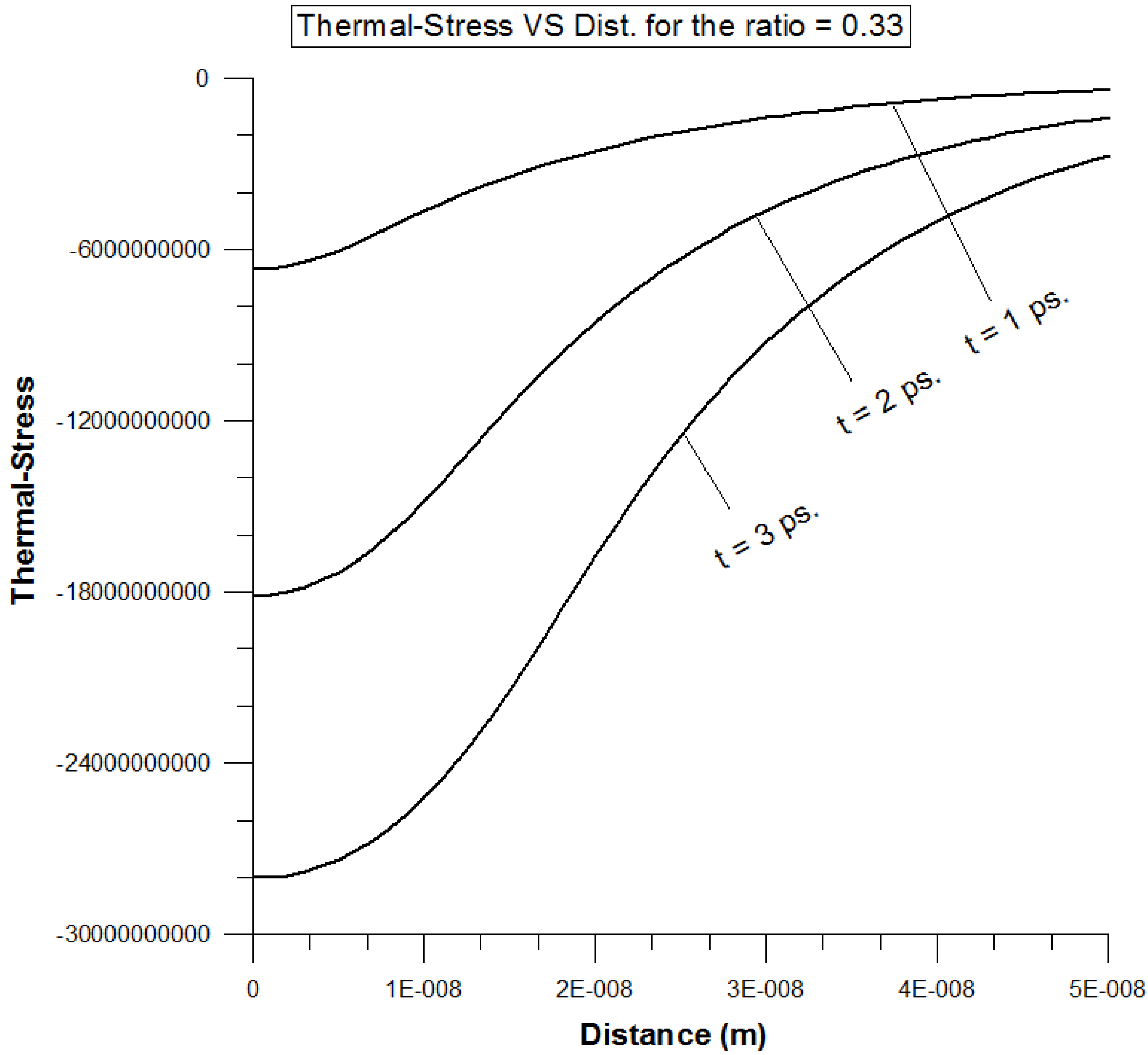

Figure 2.9: Thermal-Stress Variation inside the substrate material for different heating periods for the pulse ratio = 0.33

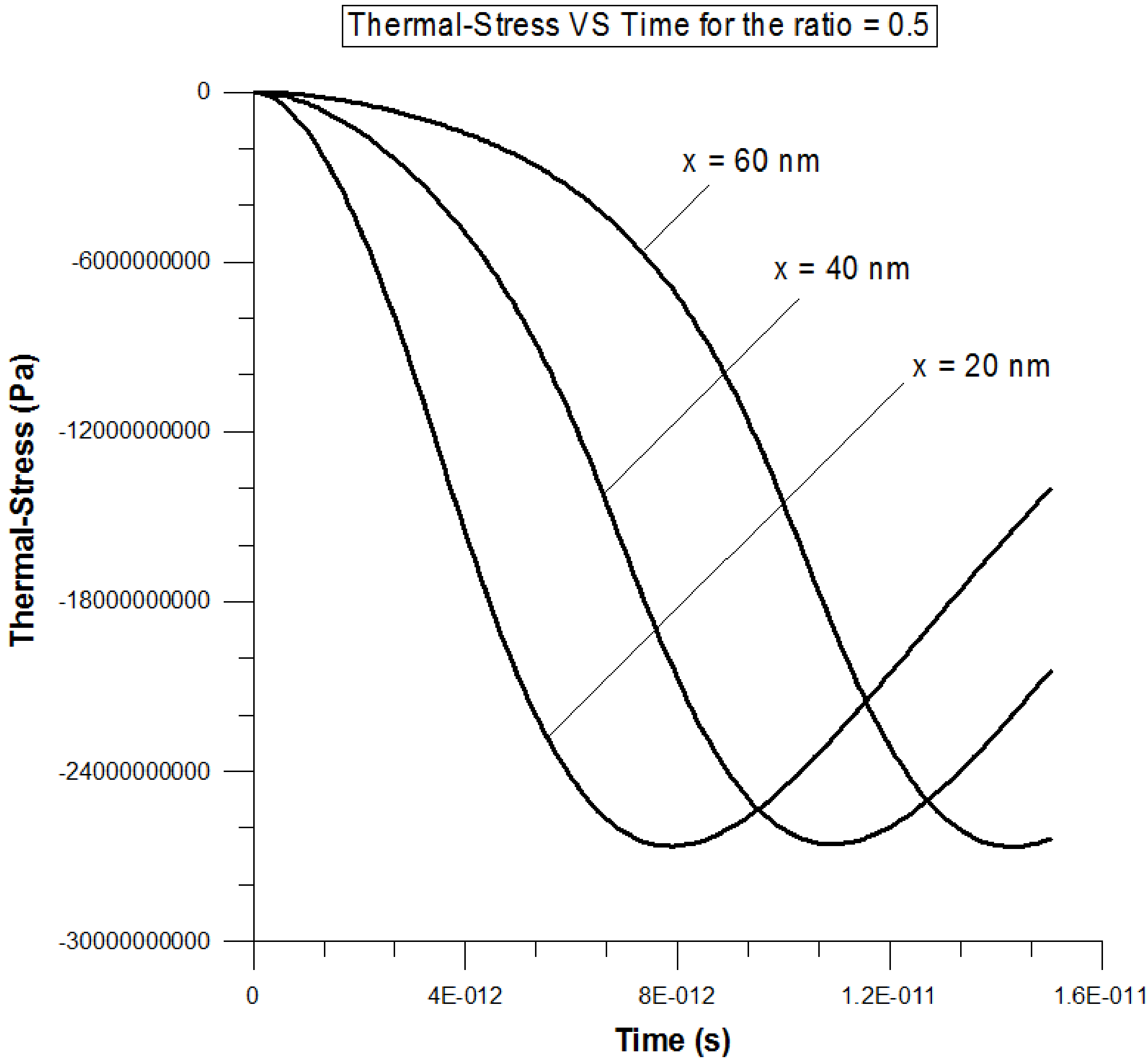

Figure 2.10: Thermal-Stress Variation with time at different depths for the pulse ratio = 0.5

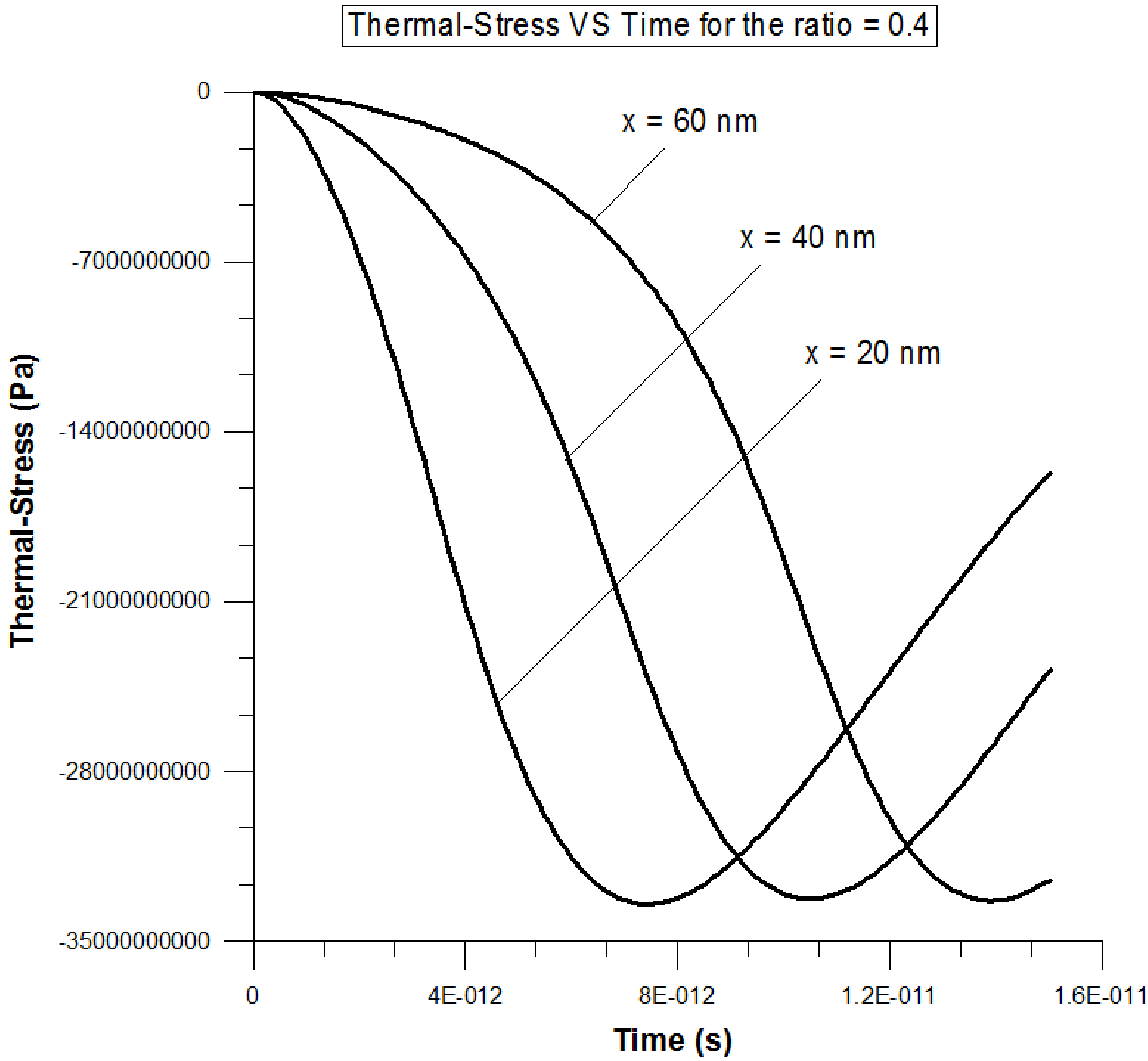

Figure 2.11: Thermal-Stress Variation with time at different depths for the pulse ratio = 0.4

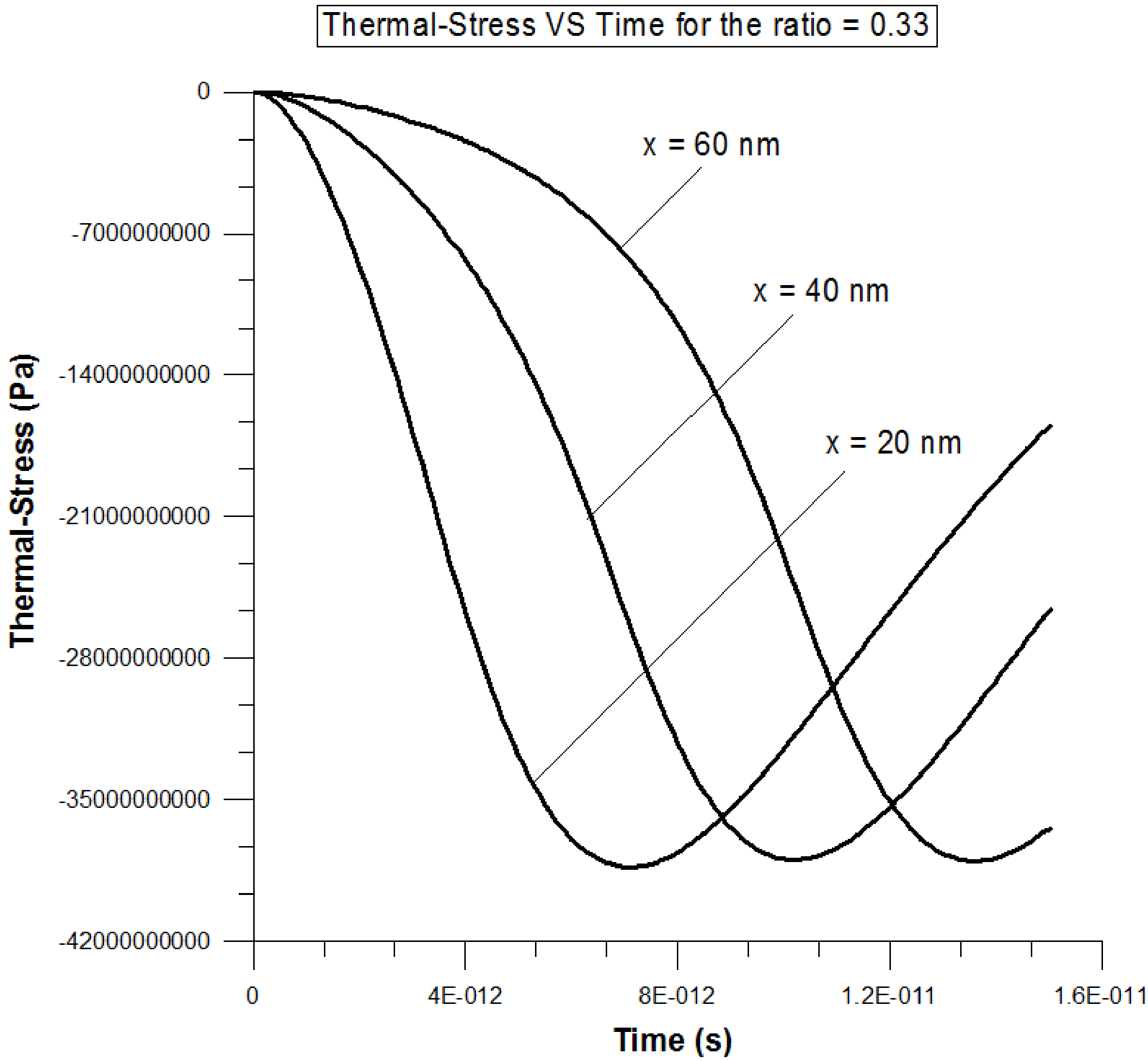

Figure 2.12: Thermal-Stress Variation with time at different depths for the pulse ratio = 0.33

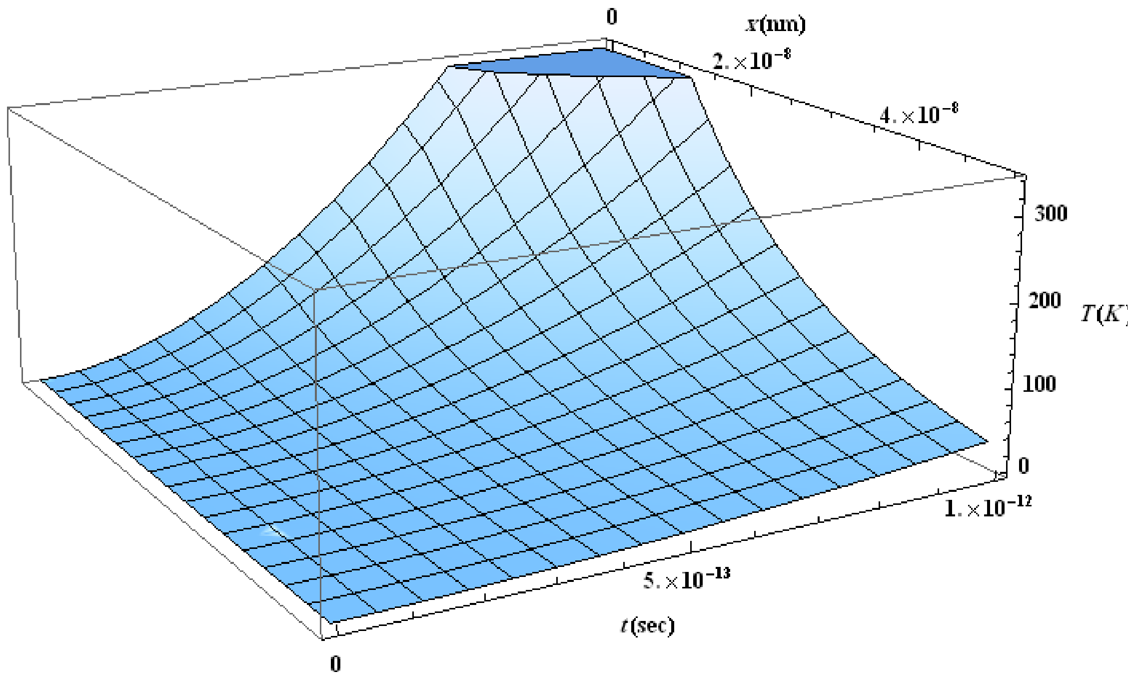

Figure 2.13: 3D-Temporal variation of temperature for the laser pulse parameter ratio = 0.5

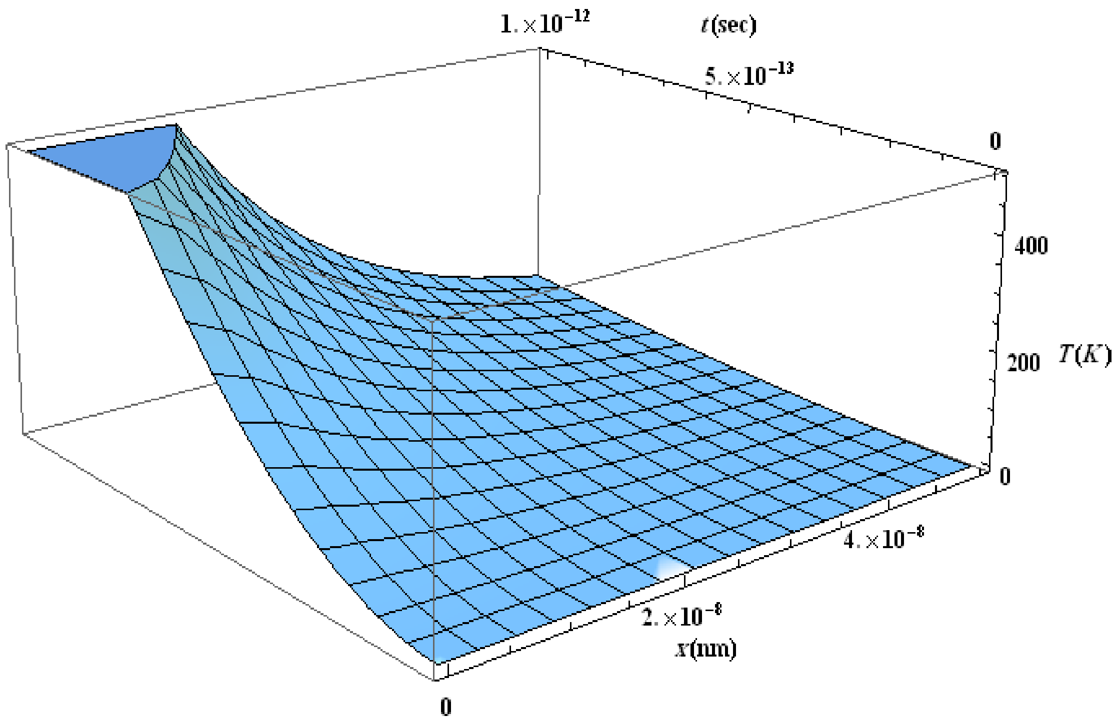

Figure 2.14: 3D-Temporal variation of temperature for the laser pulse parameter ratio = 0.4

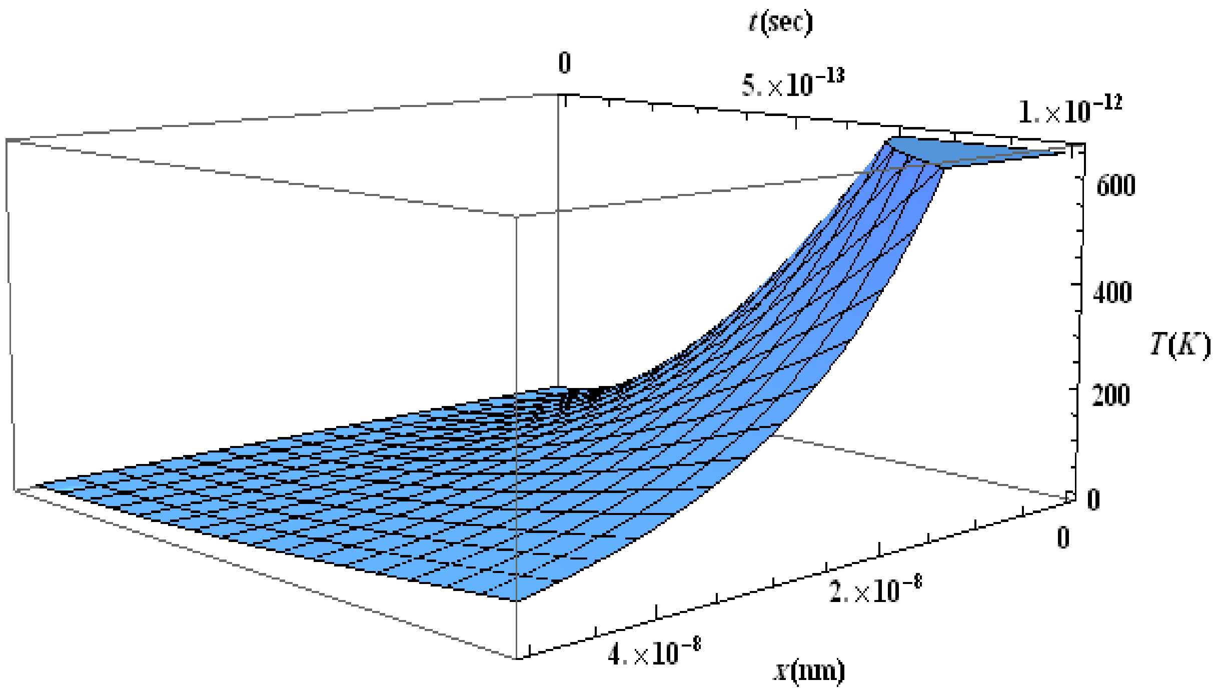

Figure 2.15: 3D-Temporal variation of temperature for the laser pulse parameter ratio = 0.33

2.3 Analytical solution of the hyperbolic heat conduction equation due to laser short-pulse heating with convective boundary condition.

2.3.1 Formulation of the Problem

Consider the laser short-pulse heating situation where the volumetric source resembling the laser pulse is incorporated with convective boundary condition

The governing hyperbolic heat conduction equation [20] can be written as,

$$A\frac{\partial^2}{\partial t^2}T(x,t) + B\frac{\partial}{\partial t}T(x,t) = \frac{\partial^2}{\partial x^2}T(x,t) + s(x,t), \tag{2.40}$$

where,

$s(x,t)$ is the laser heat source given by:

$s(x,t) = I_0\delta e^{-\delta x}\left(e^{-\beta t} - e^{-\gamma t}\right),$

$A = \frac{\rho C_p}{k\tau}$, here, ρ is the density, C_p is the specific heat, k is the thermal conductivity.

$B = \frac{\rho C_p}{k}$.

The initial and boundary conditions are given by:

$$\begin{cases} \text{Initial conditions} \begin{cases} T(x,0) = 0, \\ \frac{\partial}{\partial t}T(x,0) = 0. \end{cases} & (2.41) \\ \text{Boundary conditions} \begin{cases} \frac{\partial}{\partial x}T(0,t) = \frac{h}{k}(T(0,t) - T_0), \\ \lim_{x\to\infty} T(x,t) = 0. \end{cases} & (2.42) \end{cases}$$

In Eq. (2.41), initially, it is assumed that temperature is zero inside the substrate material and time derivative of temperature is also zero. W/ (m^2•K)

In Eq. (2.42) it is assumed that at large depth below the surface substrate material remains at the initial temperature. However, a convection boundary is assumed at the surface during the short heating period.

Let $L = \left(A\frac{\partial^2}{\partial t^2} + B\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)$ then Eq. (2.40) becomes

$$LT = s(x,t). \tag{2.43}$$

The above Equation (2.43) can be split into two equations:

$$Lu = I_0 \delta e^{-\delta x} e^{-\beta t}, \tag{2.54}$$

$$Lv = I_0 \delta e^{-\delta x} e^{-\gamma t}. \tag{2.55}$$

where u and v satisfy the same initial and boundary conditions (2.41) and (2.43).

Since L is a linear operator, the super-position principle can be applied to get the solution of Eq. (2.43).

i.e. $L(u-v) = Lu - Lv = LT$

hence, $T(x,t) = u(x,t) - v(x,t)$ (2.46)

2.3.2 Solution of the Problem

Define the Fourier cosine transform of $u(x,t)$ as :

$$U(s,t) = \int_0^\infty u(x,t)\cos(sx)\,dx$$

and the inverse Fourier cosine by

$$u(x,t) = \frac{2}{\pi}\int_0^\infty U(s,t)\cos(sx)\,ds.$$

So that applying the Fourier cosine transform in x to Eqn. (2.45) and taking into account the vanishing of temperature and its derivative for large x, we get

$$A\frac{d^2}{dt^2}U(s,t) + B\frac{d}{dt}U(s,t) = -s^2U(s,t) - u_x(0,t) + \frac{I_0\delta e^{-\beta t}}{s^2+\delta^2}. \qquad (2.47)$$

Applying the transformed boundary conditions for Eq. (2.47) and using

$\frac{\partial}{\partial x}u(0,t) = \frac{h}{k}(u(0,t) - T_0)$, we obtain,

$$A\frac{d^2}{dt^2}U(s,t) + B\frac{d}{dt}U(s,t) = -s^2U(s,t) - \frac{h}{k}(u(0,t) - T_0) + \frac{I_0\delta e^{-\beta t}}{s^2+\delta^2}. \qquad (2.48)$$

Eqn. (2.48) can be solved by applying the Laplace transform with respect to t

Define $L\{U(s,t)\} = U^*(s,p) = \int_0^\infty U(s,t)e^{-pt}dt$, so that

$$Ap^2U^*(s,p) + BpU^*(s,p) - AU_t(s,0)$$

$$-U(s,0)(Ap+B) = -s^2U^*(s,p) - \mathcal{L}\left\{\frac{h}{k}(u(0,t) - T_0)\right\} + \frac{I_0\delta}{(p+\beta)(s^2+\delta^2)}, \qquad (2.49)$$

where p is the Laplace transform parameter.

Assuming that $u(0,t) = T_0\, e^{-\tau\, t}$,

where,

T_0 is to be specified and

τ is given by,

$\frac{T_{max}}{T_0} = e^{\tau\, t_{peak}} \Rightarrow \tau = \frac{1}{t_{peak}}\, ln\left[\frac{T_{max}}{T_0}\right]$, where

$T_{max} = 300\ K$ and t_{peak} is the time at which $I_0\left(e^{-\beta t} - e^{-\gamma t}\right)$ attains its maximum.

therefore,

$$\mathcal{L}\left\{\frac{h\,T_0}{k}(e^{-\tau\, t} - 1)\right\} = \frac{-\tau\, h\, T_0}{k\, p\,(p+\tau)}$$

Now, applying the transformed initial conditions to Eq. (2.49) we get:

$$Ap^2U^*(s,p) + BpU^*(s,p) = -s^2U^*(s,p) - \frac{\tau\, h\, T_0}{k\, p\, (p+\tau)} + \frac{I_0\delta}{(p+\beta)(s^2+\delta^2)}. \tag{2.50}$$

Hence, $\bar{u}^*(s,p)$ is given by:

$$U^*(s,p) = \frac{1}{(Ap^2+Bp+s^2)}\left[\frac{I_0\delta}{(p+\beta)(s^2+\delta^2)} - \frac{\tau\, h\, T_0}{k\, p\, (p+\tau)}\right]. \tag{2.51}$$

Eq. (2.51) is in the Laplace and Fourier cosine domain and to get the solution in time and space domain we need to take the inverse Laplace and inverse Fourier cosine transforms.

Equation (2.51) can be written as,

$$U^*(s,p) = \frac{I_0\delta}{(p+\beta)(s^2+\delta^2)(Ap^2+Bp+s^2)} - \frac{\tau\, h\, T_0}{k\, p\, (p+\tau)(Ap^2+Bp+s^2)}. \tag{2.51}$$

Applying Laplace inverse Transform on $U^*(s,p)$ we get:

$$U(s,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} U^*(s,p)\, e^{pt}\, dp. \tag{2.53}$$

where, c is a positive real number which is chosen in such a way that all the real parts of the poles of the integrand are smaller than c.

This integral in (2.53) can be evaluated using the residue theorem as follows:

Let,

$$\mathcal{F}(p) = \frac{I_0\delta\ e^{pt}}{(p+\beta)(s^2+\delta^2)(Ap^2+Bp+s^2)} - \frac{\tau\, h\, T_0\ e^{pt}}{k\, p\, (p+\tau)(Ap^2+Bp+s^2)},$$

where, $\mathcal{F}$ has five simple poles as follows :

$P_1 = 0$,$P_2 = -\beta$, $P_3 = \tau$, $P_4 = \frac{-B+\sqrt{B^2-4As^2}}{2A}$ and $P_5 = \frac{-B-\sqrt{B^2-4As^2}}{2A}$

Equation (2.53) becomes:

$$U(s,t) = \sum_{j=1}^{5} Res\{\mathcal{F}(p), P_j\}. \tag{2.54}$$

The residues can be evaluated as follows:

$Res\{\mathcal{F}(p), P_1\} = \lim_{p\to 0}\ (p-0)\,\mathcal{F}(p) = \frac{hT_0}{k\,s^2},$

$Res\{\mathcal{F}(p), P_2\} = \lim_{p\to -\beta}\ (p+\beta)\,\mathcal{F}(p) = \frac{e^{-t\beta}\delta\,I_0}{(s^2-B\beta+A\beta^2)(s^2+\delta^2)},$

$Res\{\mathcal{F}(p), P_3\} = \lim_{p\to \tau}\ (p-\tau)\,\mathcal{F}(p) = -\frac{h\,T_0 e^{t\tau}}{k(s^2+B\tau+A\,\tau^2)},$

$Res\{\mathcal{F}(p), P_4\} = \lim_{p\to P_4}\ (p-P_4)\,\mathcal{F}(p) = \frac{-\,M_1(s,t)}{M_2(s,t)},$

where:

$M_1(s,t) =$

$$4Ae^{\left(-\frac{Bt}{2A}+\frac{\sqrt{B^2-4As^2}}{2A}t\right)}\left[\begin{array}{c} -B^2 I_0\,k\delta + B\left(-AT_0h(s^2+\delta^2)\tau + I_0k\delta\left(\sqrt{B^2-4As^2}-A\tau\right)\right)+ \\ A\left(T_0h\left(\sqrt{B^2-4As^2}+2A\beta\right)(s^2+\delta^2)\tau + I_0k\delta\left(2s^2+\sqrt{B^2-4As^2}\tau\right)\right) \end{array}\right]$$

and

$M_2(s,t) =$

$$\left[\begin{array}{c} k\sqrt{B^2-4As^2}\left(-B+\sqrt{B^2-4As^2}\right) \\ \left(-B+\sqrt{B^2-4As^2}+2A\beta\right)(s^2+\delta^2)\left(-B+\sqrt{B^2-4As^2}-2A\tau\right) \end{array}\right].$$

$Res\{\mathcal{F}(p), P_5\} = \lim_{p\to P_5}\ (p-P_5)\,\mathcal{F}(p) = \frac{N_1(s,t)}{N_2(s,t)},$

where:

$N_1(s,t) =$

$$4A\,e^{\left(-\frac{Bt}{2A}-\frac{\sqrt{B^2-4As^2}}{2A}t\right)}\left[\begin{array}{c} B^2 I_0k\delta + B\left(AT_0h(s^2+\delta^2)\tau + I_0k\delta\left(\sqrt{B^2-4As^2}+A\tau\right)\right)+ \\ A\left(T_0h\left(\sqrt{B^2-4As^2}-2A\beta\right)(s^2+\delta^2)\tau + I_0k\delta\left(-2s^2+\sqrt{B^2-4As^2}\tau\right)\right) \end{array}\right]$$

and

$N_2(s,t) =$

$$\left[\frac{k\sqrt{B^2-4As^2}\left(B+\sqrt{B^2-4As^2}\right)}{\left(B+\sqrt{B^2-4As^2}-2A\beta\right)(s^2+\delta^2)\left(B+\sqrt{B^2-4As^2}+2A\tau\right)}\right].$$

Hence $U(s,t)$ is given by :

$$U(s,t) = \left[\frac{hT_0}{k\,s^2} + \frac{e^{-t\beta}\delta\, I_0}{(s^2-B\beta+A\beta^2)(s^2+\delta^2)} - \frac{h\,T_0 e^{t\tau}}{k(s^2+B\tau+A\,\tau^2)} - \frac{M_1(s,t)}{M_2(s,t)} + \frac{N_1(s,t)}{N_2(s,t)}\right]. \tag{2.55}$$

Equation (2.55) is simplified to :

$U(s,t) =$

$$e^{-\left[\frac{\left(B+\sqrt{B^2-4As^2}+2A\beta\right)}{2A}\right]t} \times$$

$$\left\{ \frac{\left(\begin{array}{c} I_0 k s^2\, \delta\left(s^2+\tau(B+A\tau)\right) \left(\begin{array}{c} Be^{t\beta}\left(-1+e^{\frac{\sqrt{B^2-4As^2}\,t}{A}}\right) - 2e^{\frac{\left(B+\sqrt{B^2-4As^2}\right)t}{2A}}\sqrt{B^2-4As^2} + \\ e^{t\left(\frac{\sqrt{B^2-4As^2}}{A}+\beta\right)}\left(\sqrt{B^2-4As^2}-2A\beta\right) + \\ e^{t\beta}\left(\sqrt{B^2-4As^2}+2A\beta\right) \end{array} \right) + \\ T_0 e^{\beta\, t} h\left(s^2+\beta(-B+A\beta)\right)\left(s^2+\delta^2\right) \left(\begin{array}{c} 2\, s^2\sqrt{B^2-4As^2}\; e^{\left(\frac{t\left(B+\sqrt{B^2-4As^2}+2A\tau\right)}{2A}\right)} + \\ \tau\left(-B^2+2As^2+B\sqrt{B^2-4As^2}-AB\tau+A\sqrt{B^2-4As^2}\,\tau\right) - \\ 2e^{\frac{\left(B+\sqrt{B^2-4As^2}\right)t}{2A}}\sqrt{B^2-4As^2}\left(s^2+\tau(B+A\tau)\right) + \\ e^{\frac{\sqrt{B^2-4As^2}\,t}{A}}\tau\left(B^2-2As^2+A\sqrt{B^2-4As^2}\,\tau+B\left(\sqrt{B^2-4As^2}+A\tau\right)\right) \end{array} \right) \end{array} \right)}{\left(2ks^2\sqrt{B^2-4As^2}\left(-s^2+\beta(B-A\beta)\right)\left(s^2+\delta^2\right)\left(s^2+\tau(B+A\tau)\right)\right)} \right\}.$$

Now, we find $u(x,t)$ by taking the inverse Fourier cosine transform of Eq. (2.55),

i.e.

$$u(x,t) = \frac{2}{\pi}\int_0^\infty U(s,t)\cos(sx)\, ds \tag{2.56}$$

The function $U(s,t)$ has three non-negative singularities at

$s_1 = 0$, provided that $\leq \frac{B}{A}$, $s_2 = \sqrt{B\beta - A\beta^2}$ and $s_3 = \frac{B}{2\sqrt{A}}$ .However, an investigation, (see appendix) of Eq. (2.55) reveals that the function $\bar{u}(s,t)$ has a removable singularities at $s_1 = 0$ and at $s_2 = \sqrt{B\beta - A\beta^2}$. Also, a graphical representation of the integrand shows that the integrand oscillates for $s_2 \geq \frac{B}{2\sqrt{A}}$ and thus the contribution to the integral vanishes due to self-cancellation effect for $s_2 \geq \frac{B}{2\sqrt{A}}$.

We can therefore restrict the range and write: $u(x,t) = \frac{2}{\pi}\int_0^{\frac{B}{2\sqrt{A}}} U(s,t)\cos(sx)\,ds.$

Similarly, for Eq. (2.45) we have $Lv = I_0\delta e^{-\delta x}e^{-\gamma t}$ which can be solved by the same method above to obtain the following:

$$v(x,t) = \frac{2}{\pi}\int_0^{\frac{B}{2\sqrt{A}}} V(s,t)\cos(sx)\,ds, \tag{2.57}$$

where, $V(s,t)$ can be found by replacing β by γ in Eq. (2.55).

Therefore, the final solution of this problem is given by Eq. (2.46) which is

$$T(x,t) = u(x,t) - v(x,t)$$

$$= \frac{2}{\pi}\int_0^{\frac{B}{2\sqrt{A}}} U(s,t)\cos(sx)\,ds - \frac{2}{\pi}\int_0^{\frac{B}{2\sqrt{A}}} V(s,t)\cos(sx)\,ds. \tag{2.58}$$

These integrals in (2.58) have been evaluated numerically using Mathematica to obtain the value of $T(x,t)$ at given (x,t). These results are given in Figures 2.16 to 2.21 below.

2.3.3 Thermal-Stress Study

In this part, we study the thermal stress influence on this heating process due to the full laser pulse volumetric heat source resulting from convective boundary condition.

The equation of the thermal stress is given by the following PDE [21]:

$$\frac{\partial^2}{\partial x^2}\sigma(x,t) - \left(\frac{(1+v)(1-2\,v)\,\rho}{E\,(1-v)}\right)\frac{\partial^2}{\partial t^2}\sigma(x,t) = \left(\frac{(1+v)}{(1-v)}\,\rho\,\alpha_T\right)\frac{\partial^2}{\partial t^2}T(x,t), \tag{2.59}$$

where,

$\sqrt{\frac{(1+v)(1-2\,v)\,\rho}{E\,(1-v)}}$ is the velocity of propagation of elastic wave.

T is the temperature of the substrate material.

The initial and boundary conditions considered here are the following :

$$\begin{cases} \text{Initial conditions} \begin{cases} \sigma(x,0) = 0, \\ \frac{\partial}{\partial t}\sigma(x,0) = 0. \end{cases} & (2.60) \\ \text{Boundary conditions} \begin{cases} \frac{\partial}{\partial x}\sigma(0,t) = 0, \\ \lim_{x\to\infty}\sigma(x,t) = 0. \end{cases} & (2.61) \end{cases}$$

In Eq. (2.50), initially, it is assumed that thermal stress is zero inside the substrate material and time derivative of stress is also zero.

Equation (2.51) expresses that the stress at large distance (depth) is zero and that surface stress gradient also vanishes.

For simplicity,

let,

$m = \sqrt{\frac{(1+v)(1-2\,v)\,\rho}{E\,(1-v)}}$ and

$n = \frac{(1+v)}{(1-v)}\,\rho\,\alpha_T$

Therefore, on taking the Fourier cosine transform of equation Eq. (20), we obtain:

$$-s^2\bar{\sigma}(s,t) - \sigma_x(0,t) - m^2\frac{d^2}{dt^2}\bar{\sigma}(s,t) = n\,\frac{d^2}{dt^2}\bar{T}(s,t). \tag{2.62}$$

Applying the transformed boundary conditions for Eq. (23) we obtain

$$-s^2\bar{\sigma}(s,t) - m^2\frac{d^2}{dt^2}\bar{\sigma}(s,t) = n\,\frac{d^2}{dt^2}\bar{T}(s,t). \tag{2.63}$$

This equation can be solved by applying the Laplace transform with respect to t. On doing that on Eq. (2.63), we get:

$$-s^2\bar{\sigma}^*(s,p) - m^2p^2\,\bar{\sigma}^*(s,p) - p\,\bar{\sigma}(s,0) - \bar{\sigma}_t(s,0) = n\,p^2\bar{T}^*(s,p). \tag{2.64}$$

Now, applying the transformed initial conditions to Eq. (2.64) we get :

$$-s^2\bar{\sigma}^*(s,p) - m^2p^2\,\bar{\sigma}^*(s,p) = n\,p^2\bar{T}^*(s,p). \tag{2.65}$$

Hence, $\bar{\sigma}^*(s,p)$ is given by :

$$\bar{\sigma}^*(s,p) = \frac{-n\,p^2}{(m^2p^2+s^2)}\,\bar{T}^*(s,p). \tag{2.66}$$

This solution (2.66) is in the transformed domain and to get the solution in time and space domain we need to take the inverse Laplace and inverse Fourier cosine transforms.

Recall that from Eq. (2.46) ,

$T(x,t) = u(x,t) - v(x,t)$, which implies,

$$\bar{T}^*(s,p) = \bar{u}^*(s,p) - \bar{v}^*(s,p)\,. \tag{2.67}$$

Substitute this into Eq. (27) we get :

$$\bar{\sigma}^*(s,p) = \left[\frac{-n\,p^2}{(m^2p^2+s^2)}\right][\,U^*(s,p) - V^*(s,p)\,]. \tag{2.68}$$

Now, let

$$\bar{\phi}^*(s,p) = \frac{-n\,p^2}{(m^2p^2+s^2)}\ U^*(s,p) \tag{2.69}$$

and

$$\bar{\psi}^*(s,p) = \frac{-n\,p^2}{(m^2p^2+s^2)}\,V^*(s,p), \tag{2.70}$$

where U^* and V^* are the functions found above in the previous part.

So that $\bar{\phi}^*(s,p)$ becomes:

$$\bar{\phi}^*(s,p) = \frac{-n\,p^2}{(m^2p^2+s^2)}\,\frac{1}{(Ap^2+Bp+s^2)}\left[\frac{I_0\delta}{(p+\beta)(s^2+\delta^2)} - \frac{\tau\,h\,T_0}{k\,p\,(p-\tau)}\right]. \tag{2.71}$$

The thermal stress can be found by:

$$\sigma(x,t) = \phi(x,t) - \psi(x,t). \tag{2.72}$$

Applying Laplace inverse Transform on $\bar{\phi}^*(s,p)$ we get :

$$\bar{\phi}(s,t) = \frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \bar{\phi}^*(s,p)\,e^{pt}\,dp, \tag{2.73}$$

where, ε is a positive real number, which is chosen in such a way that all the real parts of the poles of the integrand are smaller than ε.

This integral in (2.73) can be evaluated using the residue theorem as follows :

Let,

$$\mathcal{H}(p) = \frac{n\,p\,(I_0k\,p\,\delta\,(p-\tau) - T_0h\,(p+\beta)\,(s^2+\delta^2)\,\tau)\,e^{t\,p}}{k(Bp+Ap^2+s^2)(m^2p^2+s^2)(p+\beta)(s^2+\delta^2)\,(p-\tau)},$$

then $\mathcal{H}$ has six simple poles as follows :

$P_1 = \tau$, $P_2 = -\beta$, $P_3 = \frac{i\,s}{m}$, $P_4 = \frac{-i\,s}{m}$, $P_5 = \frac{-B+\sqrt{B^2-4As^2}}{2A}$ and $P_6 = \frac{-B-\sqrt{B^2-4As^2}}{2A}$

so that,

$$\bar{\phi}(s,t) = \sum_{i=1}^{5} Res[\mathcal{H}(p), P_i]. \tag{2.74}$$

The residues are evaluated as follows :

$$Res[\mathcal{H}(p), P_1] = \lim_{p\to P_1}\ (p - P_1)\ \ \mathcal{H}(p)$$

$$= -\frac{T_0 e^{t\tau} hn\tau^2}{k(s^2+B\tau+A\tau^2)(s^2+m^2\tau^2)},$$

$$Res[\mathcal{H}(p), P_2] = \lim_{p\to P_2}\ (p - P_2)\ \ \mathcal{H}(p)$$

$$= \frac{e^{-t\beta} I_0 n\beta^2\delta}{(s^2-B\beta+A\beta^2)(s^2+m^2\beta^2)(s^2+\delta^2)},$$

$$Res[\mathcal{H}(p), P_3] = \lim_{p\to P_3}\ (p - P_3)\ \ \mathcal{H}(p)$$

$$= \frac{e^{\left(\frac{ist}{m}\right)} n\left(-iI_0ks^2\delta+T_0hms^3\tau-iT_0hm^2s^2\beta\tau+I_0kms\delta\tau+T_0hms\delta^2\tau-iT_0hm^2\beta\delta^2\tau\right)}{2ks(iBm-As+m^2s)(-is-m\beta)(s^2+\delta^2)(s+im\tau)},$$

$$Res[\mathcal{H}(p), P_4] = \lim_{p\to P_4}\ (p - P_4)\ \ \mathcal{H}(p)$$

$$= \frac{e^{-\left(\frac{ist}{m}\right)} n\left(iI_0ks^2\delta+T_0hms^3\tau+iT_0hm^2s^2\beta\tau+I_0kms\delta\tau+T_0hms\delta^2\tau+iT_0hm^2\beta\delta^2\tau\right)}{2ks(-iBm-As+m^2s)(is-m\beta)(s^2+\delta^2)(s-im\tau)},$$

$$Res[\mathcal{H}(p), P_5] = \lim_{p\to P_5}\ (p - P_5)\ \ \mathcal{H}(p)$$

$$= \frac{2An\left(-B+\sqrt{B^2-4As^2}\right) e^{\left(-\frac{Bt}{2A}+\frac{\sqrt{B^2-4As^2}\,t}{2A}\right)}\ f_1(s,t)}{f_2(s,t)},$$

where,

$f_1(s,t) =$

$$\begin{bmatrix} -B^2 I_0 k\delta + B\left(-AT_0 h(s^2+\delta^2)\tau + I_0 k\delta\left(\sqrt{B^2-4As^2} - A\tau\right)\right) + \\ A\left(T_0 h\left(\sqrt{B^2-4As^2} + 2A\beta\right)(s^2+\delta^2)\tau + I_0 k\delta\left(2s^2 + \sqrt{B^2-4As^2}\tau\right)\right) \end{bmatrix},$$

$f_2(s,t) =$

$$\begin{bmatrix} k\sqrt{B^2-4As^2}\left(-B^2m^2 - 2A^2s^2 + 2Am^2s^2 + Bm^2\sqrt{B^2-4As^2}\right) \\ \left(-B + \sqrt{B^2-4As^2} + 2A\beta\right)(s^2+\delta^2)\left(-B + \sqrt{B^2-4As^2} - 2A\tau\right) \end{bmatrix}.$$

$Res[\mathcal{H}(p), P_6] = \lim_{p \to P_6} \ (p - P_6) \ \ \mathcal{H}(p)$

$$= \frac{2An\left(B+\sqrt{B^2-4As^2}\right) e^{\left(-\frac{Bt}{2A} - \frac{\sqrt{B^2-4As^2}t}{2A}\right)} \ g_1(s,t)}{g_2(s,t)},$$

where,

$g_1(s,t) =$

$$\begin{bmatrix} B^2 I_0 k\delta + B\left(AT_0 h(s^2+\delta^2)\tau + I_0 k\delta\left(\sqrt{B^2-4As^2} + A\tau\right)\right) + \\ A\left(T_0 h\left(\sqrt{B^2-4As^2} - 2A\beta\right)(s^2+\delta^2)\tau + I_0 k\delta\left(-2s^2 + \sqrt{B^2-4As^2}\tau\right)\right) \end{bmatrix},$$

$g_2(s,t) =$

$$\begin{bmatrix} k\sqrt{B^2-4As^2}\left(B^2m^2 + 2A^2s^2 - 2Am^2s^2 + Bm^2\sqrt{B^2-4As^2}\right) \\ \left(B + \sqrt{B^2-4As^2} - 2A\beta\right)(s^2+\delta^2)\left(B + \sqrt{B^2-4As^2} + 2A\tau\right) \end{bmatrix},$$

$\bar{\phi}(s,t)$ becomes :

$\bar{\phi}(s,t) =$

$$\left(\begin{array}{c} -\frac{T_0 e^{t\tau} h n \tau^2}{k(s^2+B\tau+A\tau^2)(s^2+m^2\tau^2)} + \frac{e^{-t\beta} I_0 n \beta^2 \delta}{(s^2-B\beta+A\beta^2)(s^2+m^2\beta^2)(s^2+\delta^2)} + \\ \frac{e^{\left(\frac{ist}{m}\right)} n\left(-iI_0 k s^2 \delta + T_0 h m s^3 \tau - iT_0 h m^2 s^2 \beta \tau + I_0 k m s \delta \tau + T_0 h m s \delta^2 \tau - iT_0 h m^2 \beta \delta^2 \tau\right)}{2ks(iBm-As+m^2 s)(-is-m\beta)(s^2+\delta^2)(s+im\tau)} + \\ \frac{e^{-\left(\frac{ist}{m}\right)} n\left(iI_0 k s^2 \delta + T_0 h m s^3 \tau + iT_0 h m^2 s^2 \beta \tau + I_0 k m s \delta \tau + T_0 h m s \delta^2 \tau + iT_0 h m^2 \beta \delta^2 \tau\right)}{2ks(-iBm-As+m^2 s)(is-m\beta)(s^2+\delta^2)(s-im\tau)} + \\ \frac{2An\left(-B+\sqrt{B^2-4As^2}\right) e^{\left(-\frac{Bt}{2A}+\frac{\sqrt{B^2-4As^2}\,t}{2A}\right)} f_1(s,t)}{f_2(s,t)} + \\ \frac{2An\left(B+\sqrt{B^2-4As^2}\right) e^{\left(-\frac{Bt}{2A}-\frac{\sqrt{B^2-4As^2}\,t}{2A}\right)} g_1(s,t)}{g_2(s,t)} \end{array}\right). \tag{2.75}$$

The above equation can be simplified to :

$\bar{\phi}(s,t) =$

$$-\rho\alpha_T \frac{(1+v)}{(1-v)} \left\{\begin{array}{c} \frac{e^{-t\beta}\ I_0 \beta^2 \delta}{(s^2-B\beta+A\beta^2)(s^2+m^2\beta^2)(s^2+\delta^2)} - \frac{T_0 e^{t\tau} h \tau^2}{k(s^2+B\tau+A\tau^2)(s^2+m^2\tau^2)} - \\ \frac{e^{\left[-\frac{\left(B+\sqrt{B^2-4As^2}\right)t}{2A}\right]} \mathbf{K}(\mathbf{s},\mathbf{t})}{2k\sqrt{B^2-4As^2}(B^2m^2+A^2s^2-2Am^2s^2+m^4s^2)(-s^2+B\beta-A\beta^2)(s^2+\delta^2)(s^2+B\tau+A\tau^2)} + \\ \frac{\left(\begin{array}{c} s\left(\begin{array}{c} -A\left(T_0 h m^2 (s^2+m^2\beta^2)(s^2+\delta^2)\tau^2 + I_0 k s^2 \delta (s^2+m^2\tau^2)\right) + \\ m^2\left(\begin{array}{c} T_0 h \tau (s^2+m^2\beta^2)(s^2+\delta^2)(B+m^2\tau) + \\ I_0 k (s^2+B\beta)\delta(s^2+m^2\tau^2) \end{array}\right) \end{array}\right) \mathrm{Cos}\left[\frac{st}{m}\right] + \\ m\left(\begin{array}{c} B\left(T_0 h m^2 (s^2+m^2\beta^2)(s^2+\delta^2)\tau^2 + I_0 k s^2 \delta (s^2+m^2\tau^2)\right) \\ +(A-m^2)s^2\left(\begin{array}{c} T_0 h (s^2+m^2\beta^2)(s^2+\delta^2)\tau + \\ I_0 k \beta \delta (s^2+m^2\tau^2) \end{array}\right) \end{array}\right) \mathrm{Sin}\left[\frac{st}{m}\right] \end{array}\right)}{ks(B^2m^2+A^2s^2-2Am^2s^2+m^4s^2)(s^2+m^2\beta^2)(s^2+\delta^2)(s^2+m^2\tau^2)} \end{array}\right\},$$

where,

$$\mathrm{K}(s,t) = \left(\begin{array}{c} \mathrm{K}_1(s,t) - A^2\left(T_0 h \tau (s^2+\delta^2)\,\mathrm{K}_2(s,t) + I_0 k \delta\,\mathrm{K}_3(s,t)\right) - \\ A\left(\mathrm{K}_4(s,t) + B\,\mathrm{K}_5(s,t)\right) \end{array}\right),$$

where $\mathrm{K}_1(s,t)$, $\mathrm{K}_2(s,t)$, $\mathrm{K}_3(s,t)$, $\mathrm{K}_4(s,t)$ and $\mathrm{K}_5(s,t)$ are in the given in the appendix A4.

Similarly, $\bar{\psi}(s,t)$ can be found by replacing β by γ in the above equation.

Now, we find $\phi(x,t)$ by taking the inverse Fourier cosine transform of $\bar{\phi}(s,t)$

that is :

$$\phi(x,t) = \frac{2}{\pi}\int_0^\infty \bar{\phi}(s,t)\cos(sx)\,ds. \tag{2.76}$$

The function $\bar{\phi}(s,t)$ has three non-negative singularities at $sg_1 = 0$, provided that $\beta \leq \frac{B}{A}$ $sg_2 = \sqrt{B\beta - A\beta^2}$ and $sg_3 = \frac{B}{2\sqrt{A}}$. However, an investigation, see appendix, of Eq. (2.75) reveals that the singularities of function $\bar{\phi}(s,t)$ are removable. An argument similar to that in Eq. (2.54), we can then write:

$$\phi(x,t) = \frac{2}{\pi}\int_0^{\frac{B}{2\sqrt{A}}} \bar{\phi}(s,t)\cos(sx)\,ds, \tag{2.77}$$

similarly,

$$\psi(x,t) = \frac{2}{\pi}\int_0^{\frac{B}{2\sqrt{A}}} \bar{\psi}(s,t)\cos(sx)\,ds. \tag{2.78}$$

So that the thermal stress is given by

$$\sigma(x,t) = \phi(x,t) - \psi(x,t)$$

$$\frac{2}{\pi}\left(\int_0^{\frac{B}{2\sqrt{A}}} \bar{\phi}(s,t)\cos(sx)\,ds - \int_0^{\frac{B}{2\sqrt{A}}} \bar{\psi}(s,t)\cos(sx)\,ds\right). \tag{2.79}$$

These integrals have been evaluated numerically using Mathematica to obtain $\sigma(x,t)$ at given (x,t). These results are given in Figures 2.22 to 2.27 below.

2.3.4 Results and discussion

Analytical solution for the hyperbolic heat conduction equation accounting for finite speed of heat conduction with presence of the volumetric heat source is presented. The short pulse laser source is incorporated as a volumetric source in the equation and a convective boundary condition is assumed as exponential type. We have used the Laplace transform in time and the Fourier cosine transform in space to find the solution of the hyperbolic heat conduction equation incorporating the appropriate initial and boundary conditions. The inversion of the Laplace transform is analytically performed. For inversion of the Fourier cosine transform, Mathematica is used and solution displayed graphically. Furthermore, thermal stress field is obtained through coupling heat and thermal stress equations and the integral transforms are used to obtain the solutions of the coupled equations. The inversion of the Laplace transform is performed using complex residue theory while Mathematica is used for the inverse Fourier cosine transform. However, inversion from the transformed plane to the physical plane is involved with complexity. This is because of the assumption of the convective boundary condition. Also, singularities of integrand in the transformed domains. In this case, all singularities are removed and avoided through reducing the range of integration, which can be justified due to the oscillatory nature of the integral because of self-cancellation effect.

2.3.5 Tables

Table 2.3: The properties and the range of values used in the simulations.

$\alpha\ (\mathbf{m^2}/_{\mathbf{s}})$	$\mathbf{0.227 \times 10^{-4}}$
$\beta\ (\mathbf{1}/_{\mathbf{s}})$	2×10^{11}
$\delta\ (\mathbf{1}/_{\mathbf{m}})$	6.16×10^{7}
$\mathbf{I_0}\ (\mathbf{W}/_{\mathbf{m^2}})$	4×10^{20}
$\gamma\ (\mathbf{1}/_{\mathbf{s}})$	4 , 5 and 6×10^{11}
$\mathbf{A}\ (\mathbf{s^2}/_{\mathbf{m^2}})$	2.1×10^{-10}
$\mathbf{B}\ (\mathbf{s}/_{\mathbf{m^2}})$	44052.86
$\mathbf{V}$	0.3
$\mathbf{E}\ (\mathbf{1}/_{\mathbf{K}})$	200×10^{9}
$\rho\ (\mathbf{kg}/_{\mathbf{m^3}})$	7850
$\alpha_T\ (\mathbf{1}/_{\mathbf{s}})$	12×10^{-6}
$\mathbf{T}_{max}\ (\mathbf{K})$	1200
$\mathbf{T_0}\ (\mathbf{K})$	300
$\mathbf{t}_{peak}\ (\mathbf{K})$	3×10^{-12}

2.3.6 Figures

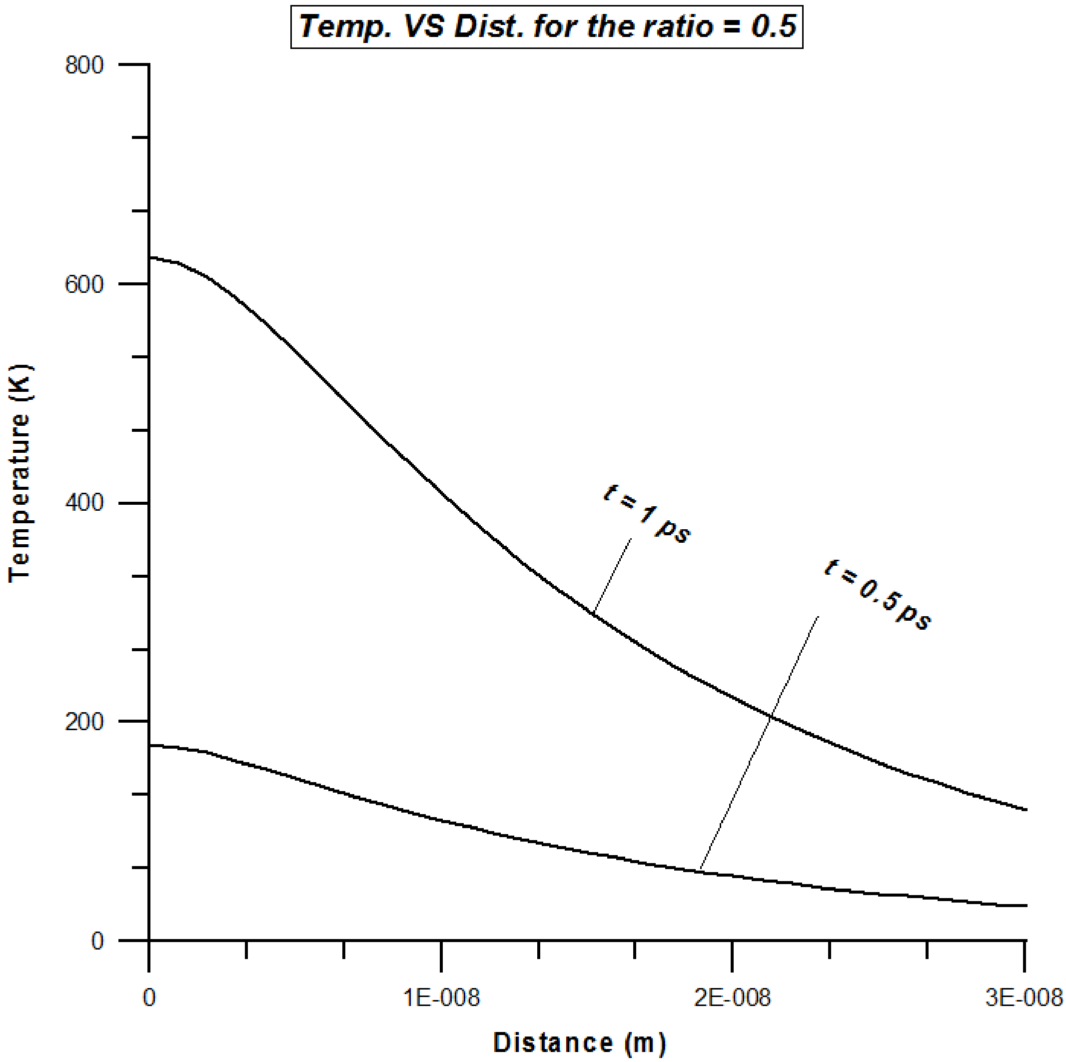

Figure 2.16: Temperature Variation inside the substrate material for different heating periods for the pulse ratio = 0.5

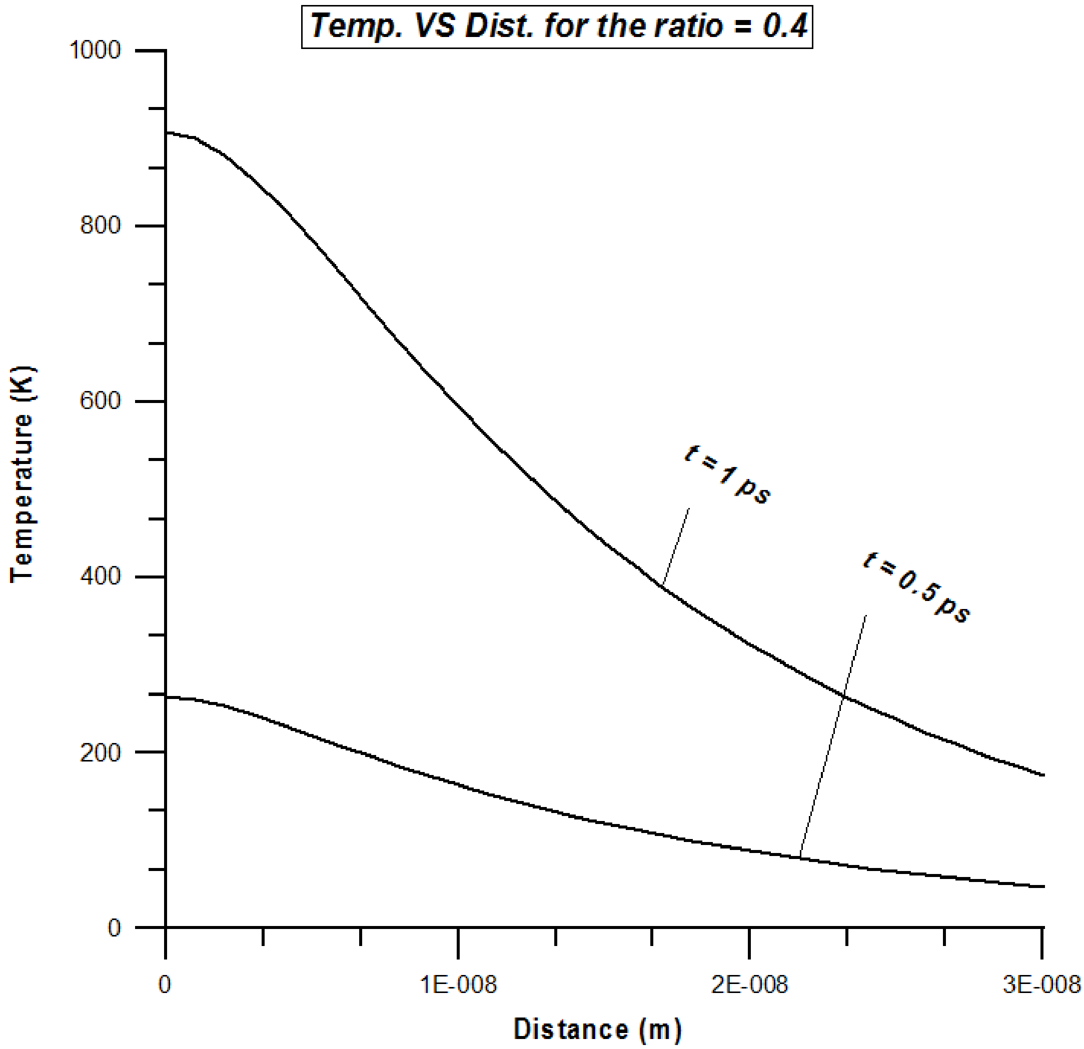

Figure 2.17: Temperature Variation inside the substrate material for different heating periods for the pulse ratio = 0.4

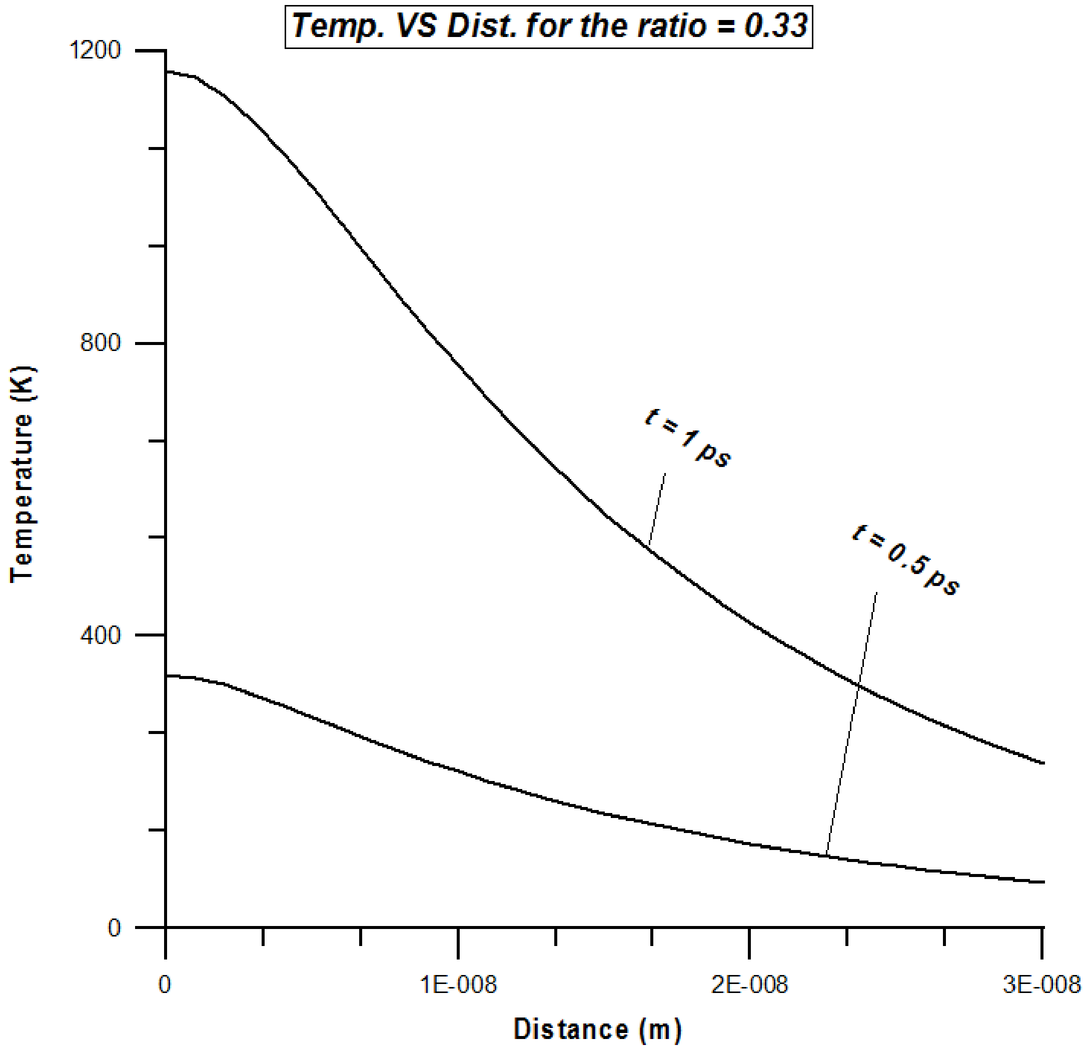

Figure 2.18: Temperature Variation inside the substrate material for different heating periods for the pulse ratio = 0.33

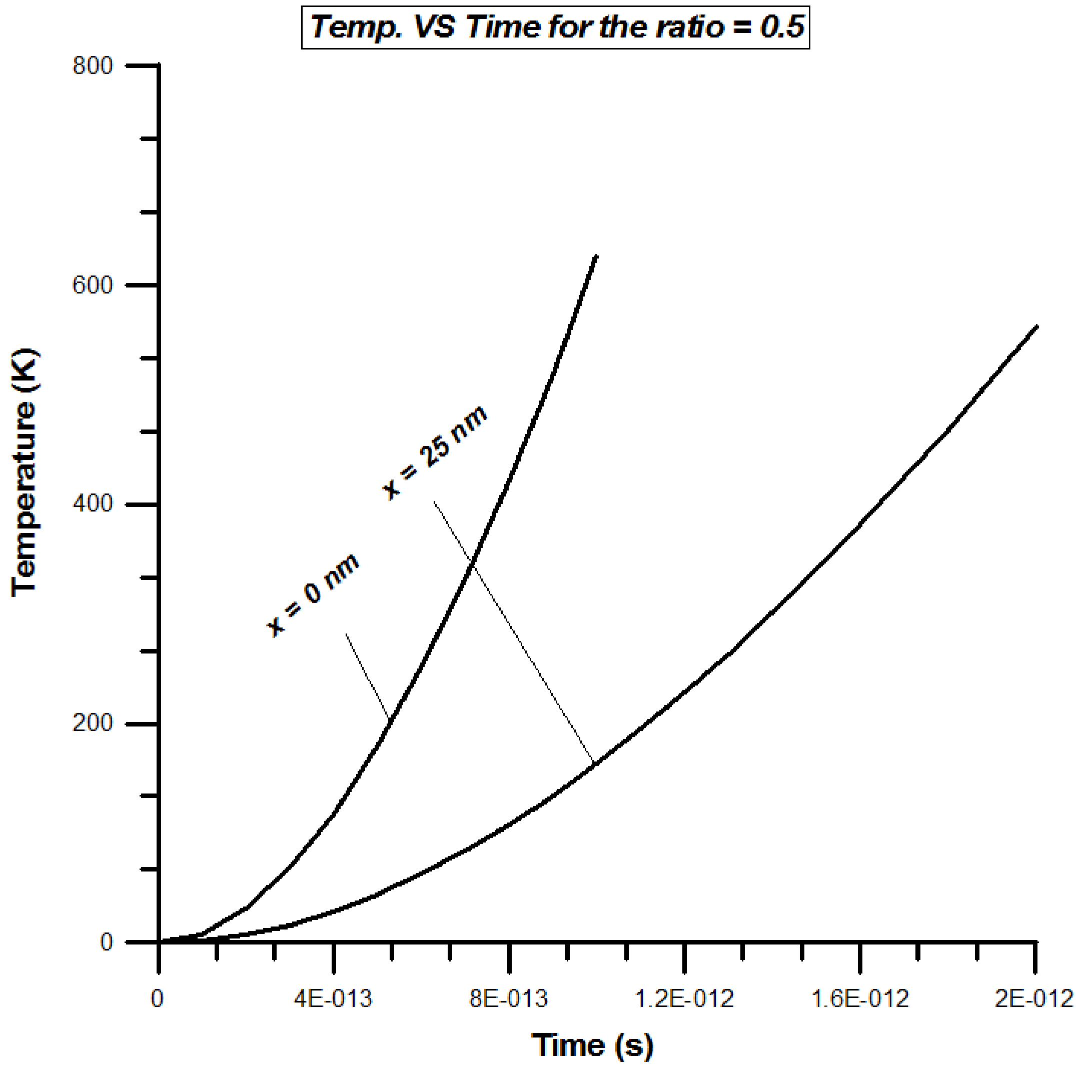

Figure 2.19: Temperature Variation with time at different depths for the pulse ratio = 0.5

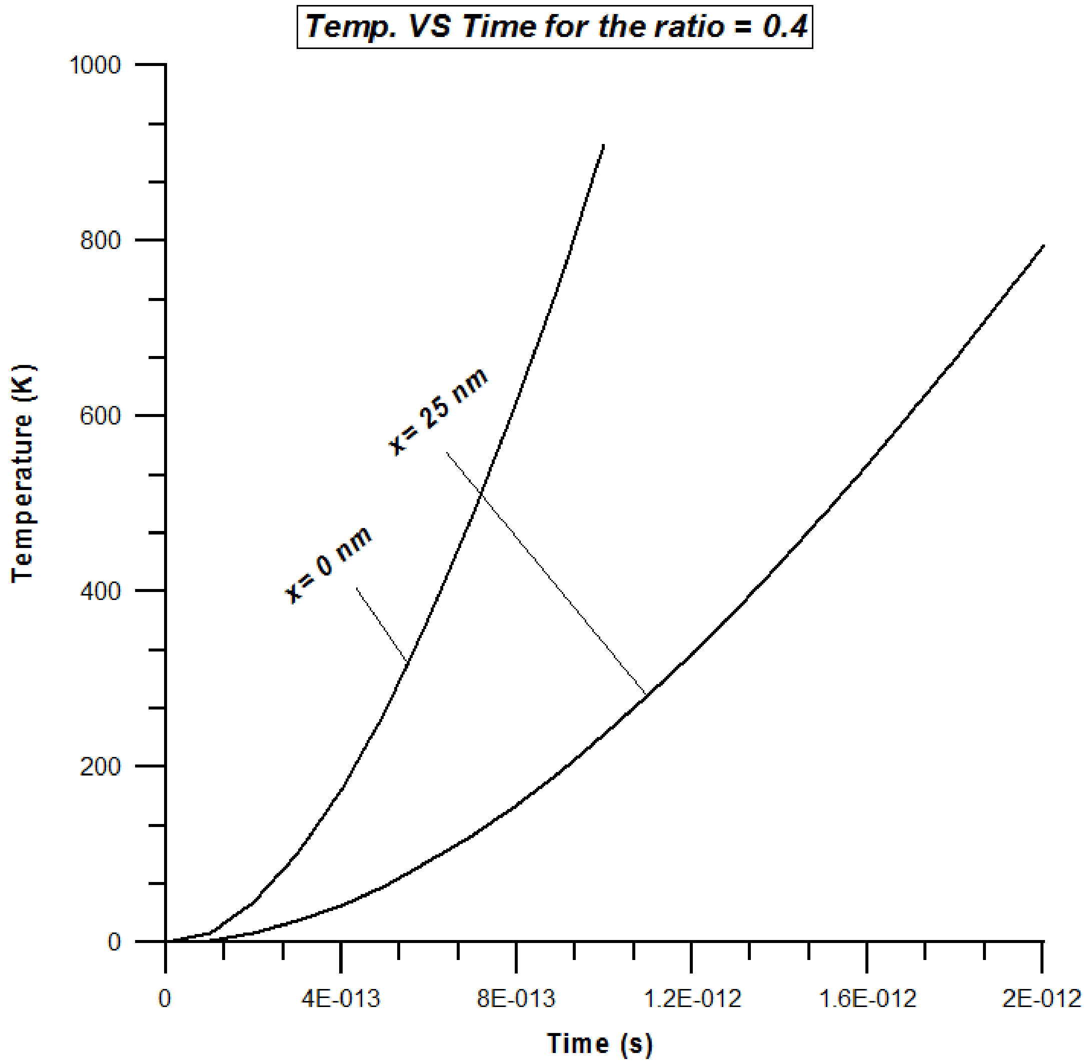

Figure 2.20: Temperature Variation with time at different depths for the pulse ratio = 0.4

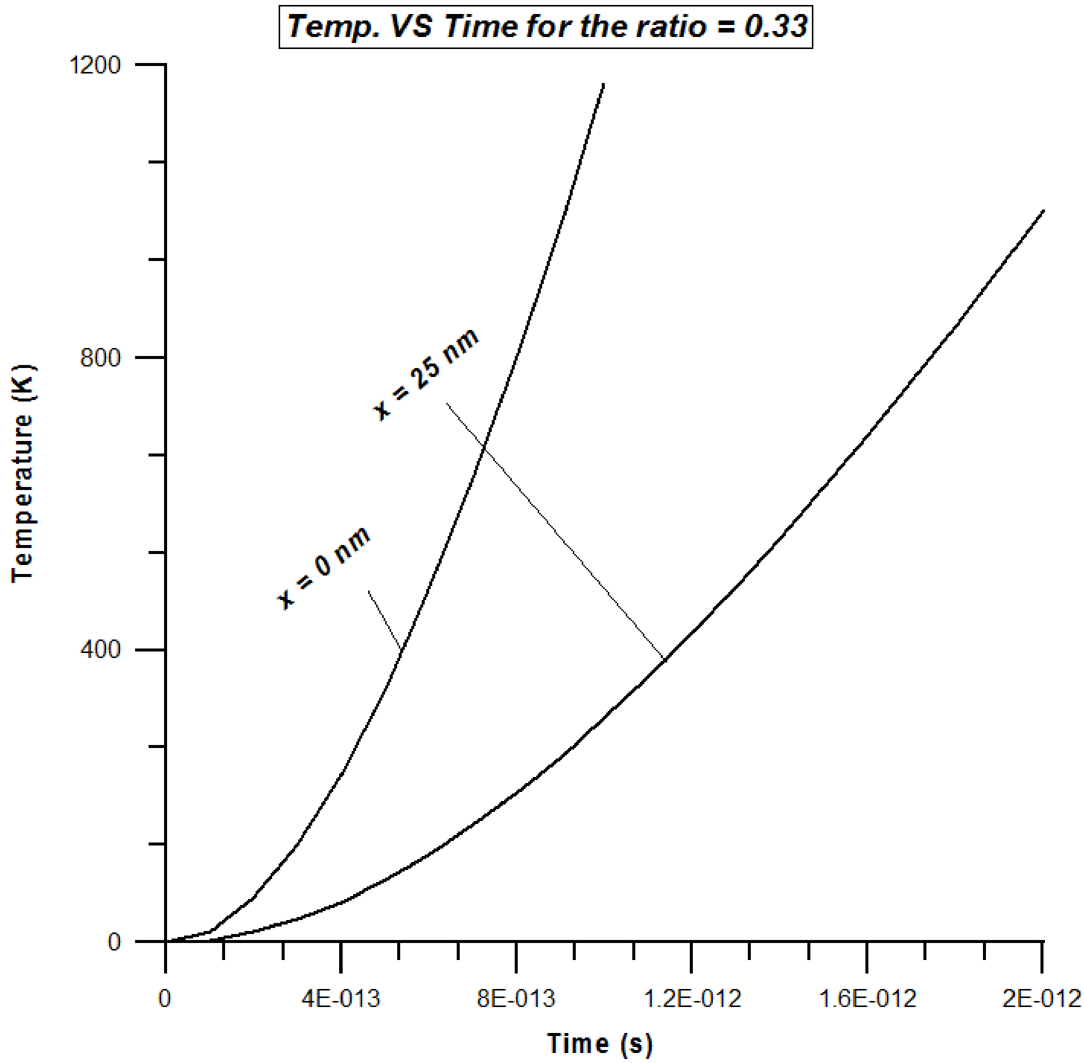

Figure 2.21: Temperature Variation with time at different depths for the pulse ratio = 0.33

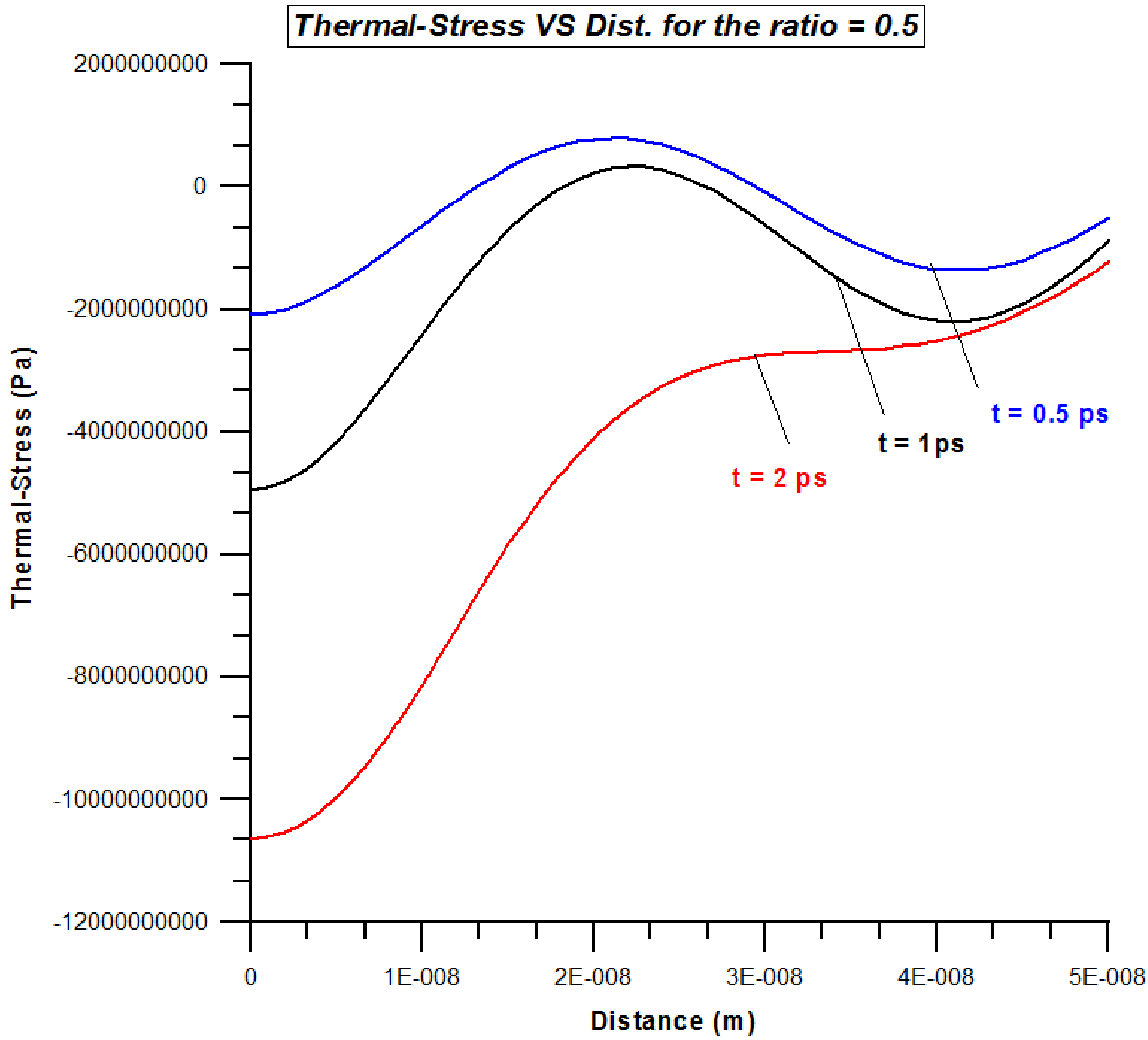

Figure 2.22: Thermal-Stress Variation inside the substrate material for different heating periods for the pulse ratio = 0.5

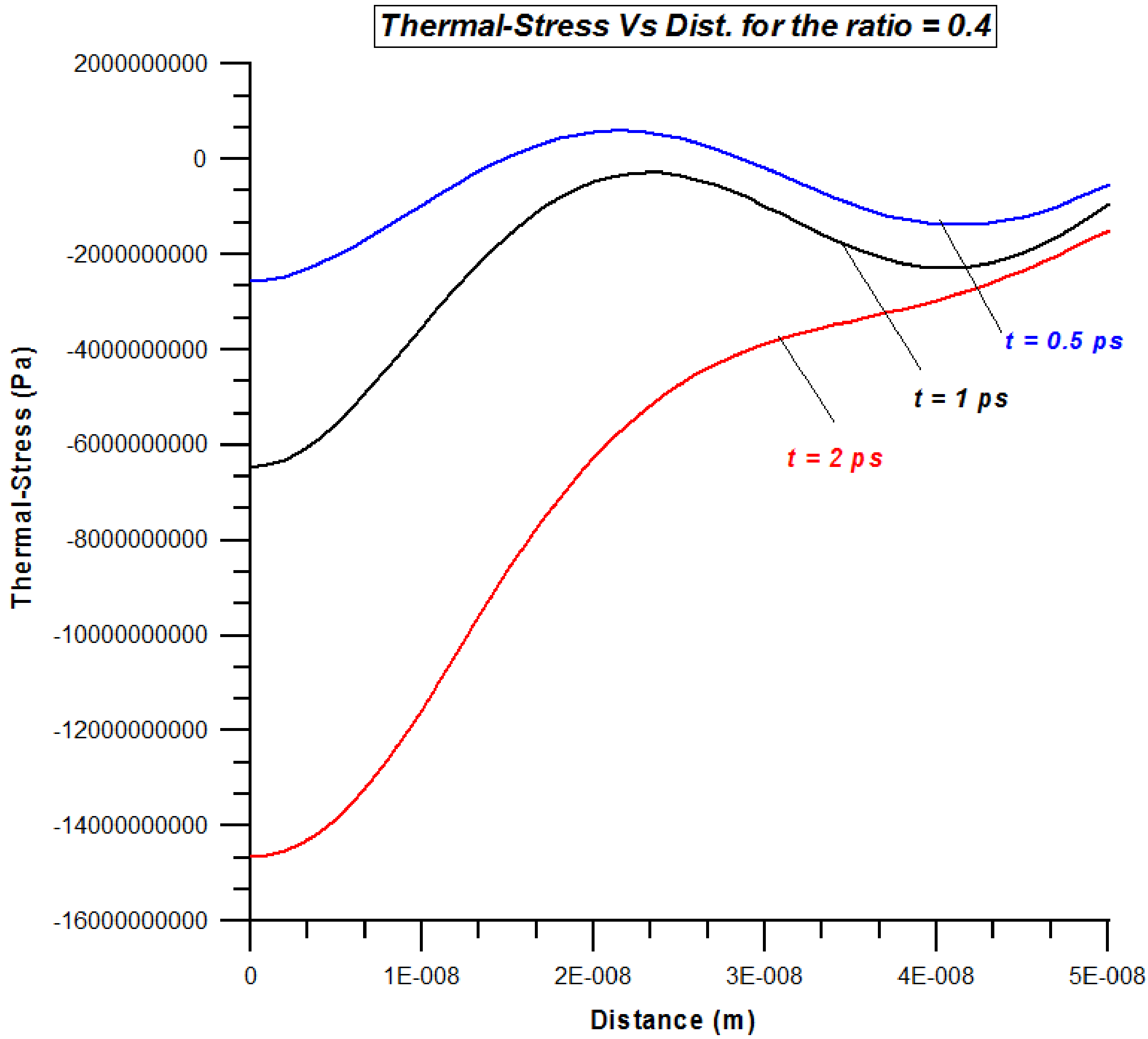

Figure 2.23: Thermal-Stress Variation inside the substrate material for different heating periods for the pulse ratio = 0.4

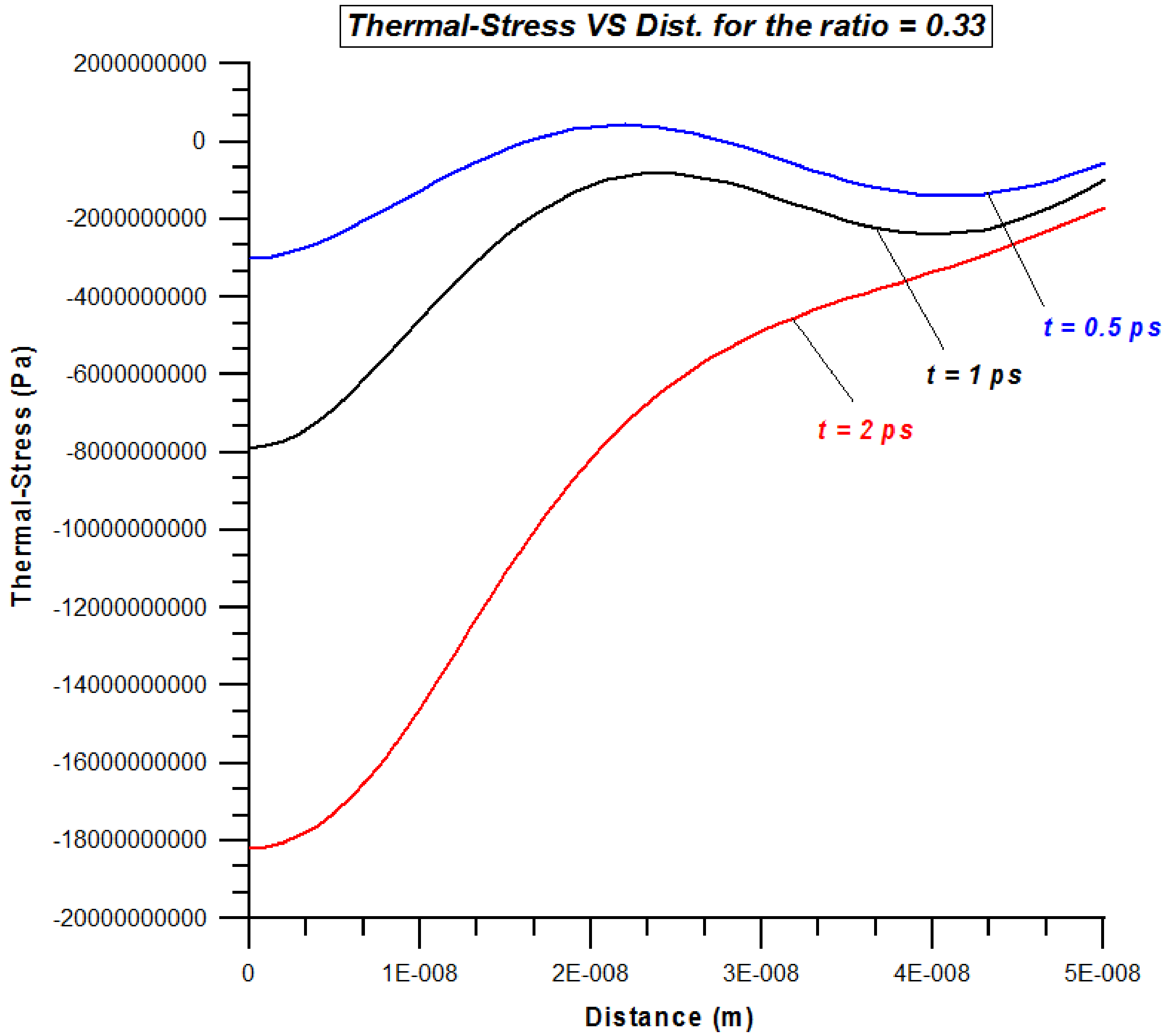

Figure 2.24: Thermal-Stress Variation inside the substrate material for different heating periods for the pulse ratio = 0.33

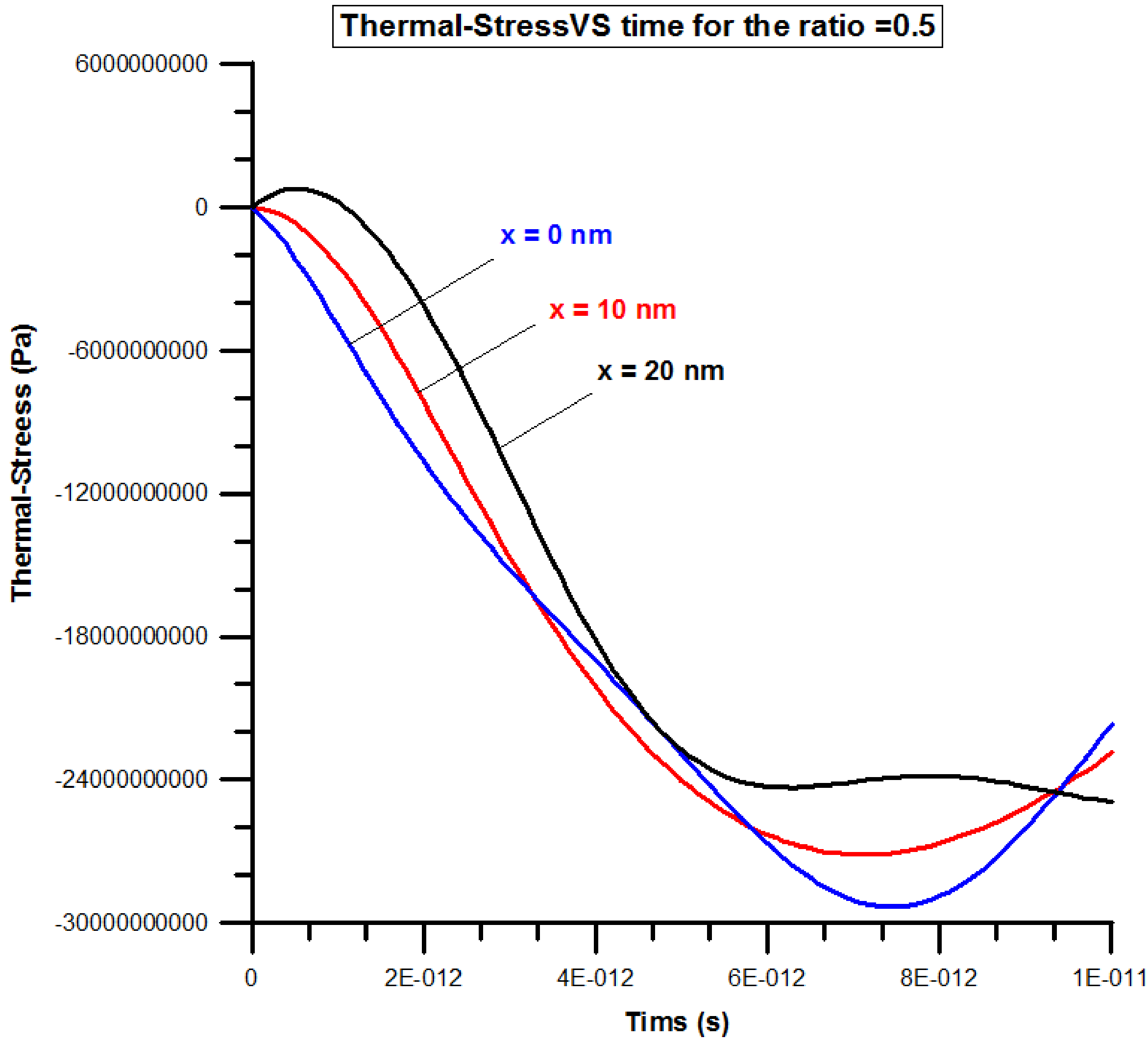

Figure 2.25: Thermal-Stress Variation with time at different depths for the pulse ratio = 0.5

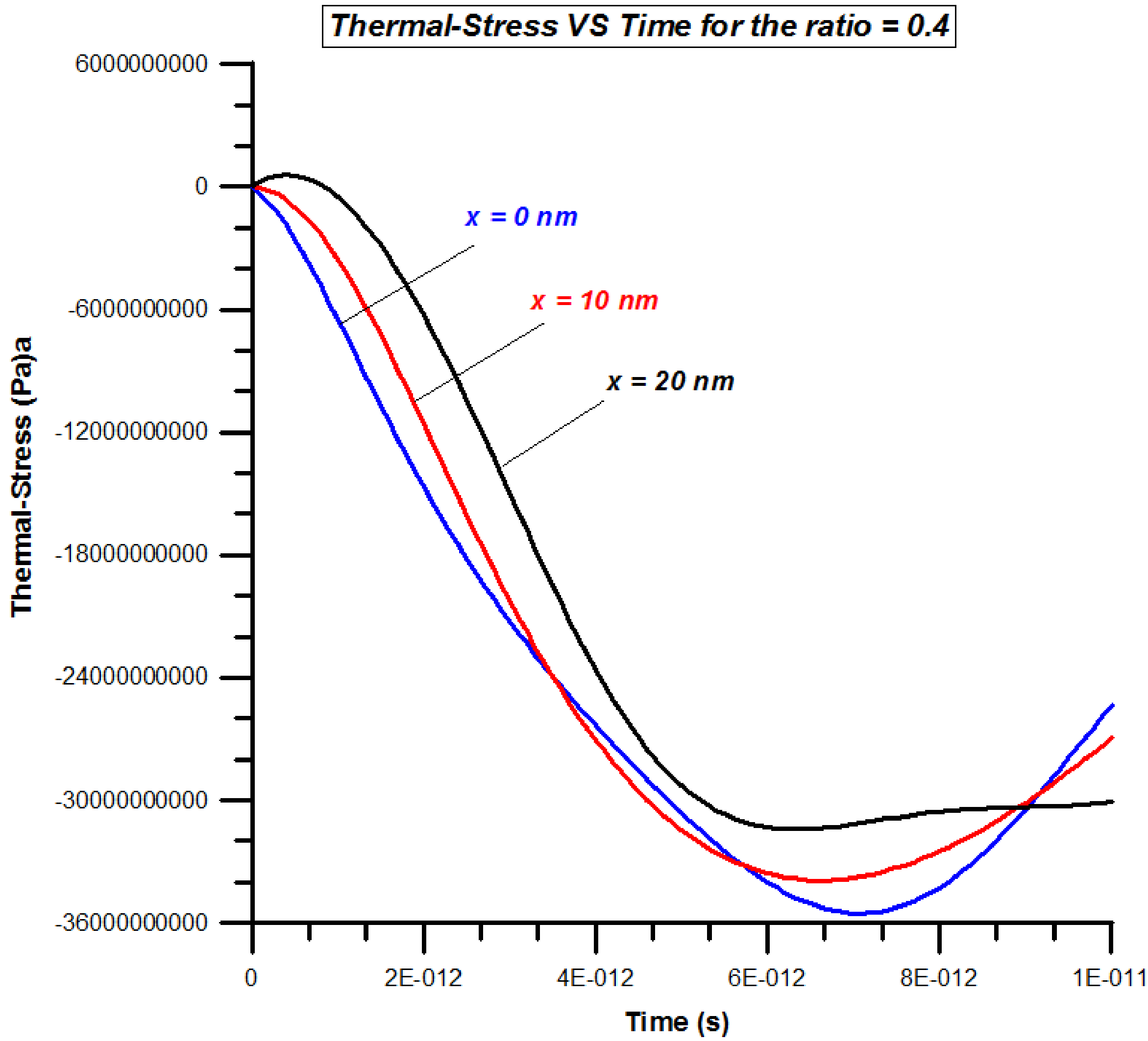

Figure 2.26: Thermal-Stress Variation with time at different depths for the pulse ratio = 0.4

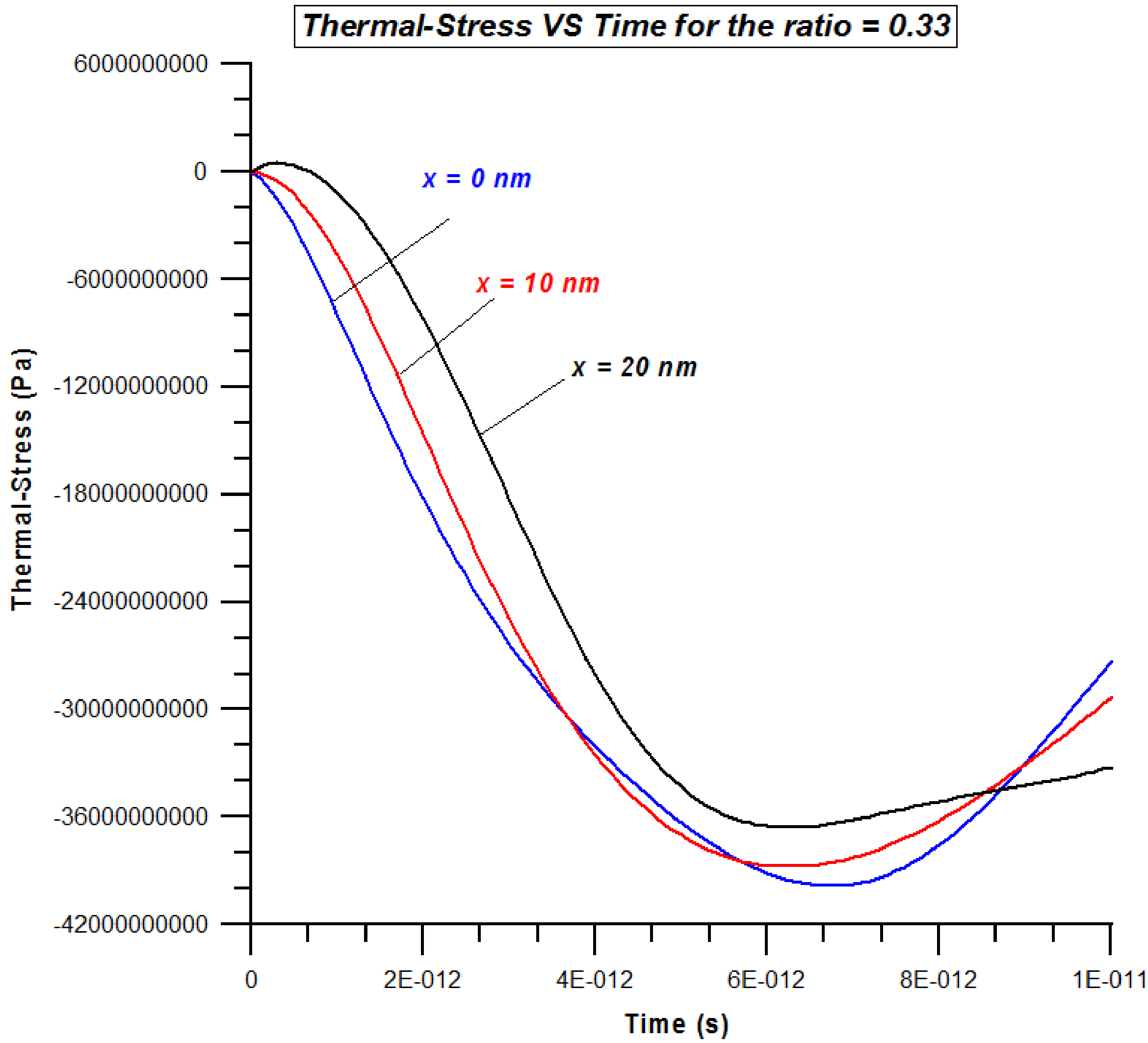

Figure 2.27: Thermal-Stress Variation with time at different depths for the pulse ratio = 0.33

Chapter 3

Determination of Temperature Distribution and Thermal Stress due to Step Input Volumetric Heat Source.

3.1 Introduction

In this chapter, we consider the hyperbolic heat conduction model considered in chapter 2 and obtain the analytical solution for the laser short-pulse heating of a solid surface. A step input volumetric source is incorporated in the analysis. Also, it is assumed that there is no convection in the boundary considered. We also study thermal stress development in the irradiated region due to the presence of this volumetric heat source. The Laplace transform in time and the Fourier cosine transform in space variable are employed to find solution of the problem in the transformation domain. The inversion of the solution from the transform plane is carried out using an analytical approach.

3.2 Determination of Temperature Distribution and Thermal Stress for Hyperbolic Heat Conduction Equation due to Step Input Volumetric Source.

3.2.1 Formulation of the Problem

Consider the laser short-pulse heating situation where the step input volumetric source resembling the laser pulse is incorporated. The governing hyperbolic heat conduction equation can be written as [20]:

$$A\frac{\partial^2}{\partial t^2}T(x,t)+B\frac{\partial}{\partial t}T(x,t)=\frac{\partial^2}{\partial x^2}T(x,t)+I_0\delta e^{-\delta x}\,(\,C_1+C_2\,)\,p(t), \tag{3.1}$$

where,

$A=\frac{\rho C_p}{k\tau}$, $B=\frac{\rho C_p}{k}$ and $p(t)=\{\ \}$. C_1 and C_2 are the laser pulse parameters.

The initial and boundary conditions are given by:

$T(x,0)=0,$

$$\frac{\partial}{\partial t}T(x,0)=0. \tag{3.2}$$

In Eq. (3.2), initially, it is assumed that temperature is zero inside the substrate material and time derivative of temperature is also zero.

$\lim_{x\to\infty}T(x,t)=0,$

$$\frac{\partial}{\partial x}T(0,t)=0. \tag{3.3}$$

In Eq. (3.3) it is assumed that at large depth below the surface substrate material remains at the initial temperature. Also, there is no convection boundary is assumed at the surface during the short heating period.

Let $D=(\,A\frac{\partial^2}{\partial t^2}+B\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\,)$, then Eq. (3.1) becomes

$$DT = I_0 \delta e^{-\delta x} \left(C_1 + C_2 \right) p(t). \tag{3.4}$$

The above Equation (3.4) can be split into two equations:

$$Du = I_0 \delta e^{-\delta x} C_1 \, p(t) \tag{3.5}$$

and

$$Dv = I_0 \delta e^{-\delta x} C_2 \, p(t). \tag{3.6}$$

where u and v satisfy the same initial and boundary conditions Eq. (3.2) and Eq. (3.3).

Since D is a linear operator, the super-position principle can be applied to get the solution of Eq. (3.4).

that is

$$D(u - v) = Du - Dv = DT,$$

hence,

$$T(x,t) = u(x,t) - v(x,t). \tag{3.7}$$

3.2.2 Solution of the Problem

Upon applying the Fourier cosine transform in x to Eq. (3.5) and taking into account the vanishing of temperature and its derivative for large x, we get

$$A \frac{d^2}{dt^2} U(s,t) + B \frac{d}{dt} U(s,t) = -s^2 U(s,t) - u_x(0,t) + \frac{I_0 \delta \, C_1 \, p(t)}{s^2 + \delta^2}. \tag{3.8}$$

Applying the boundary conditions for Eq. (3.8) we obtain

$$A \frac{d^2}{dt^2} U(s,t) + B \frac{d}{dt} U(s,t) = -s^2 U(s,t) + \frac{I_0 \delta \, C_1 \, p(t)}{s^2 + \delta^2}. \tag{3.9}$$

Eq. (3.9) can be solved by applying the Laplace transform with respect to t as follows

$$Ap^2U^*(s,p) + BpU^*(s,p) - AU_t(s,0) - U(s,0)(Ap+B) = -s^2U^*(s,p) + \frac{I_0\delta\, L\{\,C_1\,p(t)\}}{(s^2+\delta^2)}, \tag{3.10}$$

where,

$$L\{\,C_1\,p(t)\} = \frac{(e^{-\,p\mu} - e^{-p(\mu+\xi)})}{p}.$$

Now, applying the transformed initial conditions to Eq. (3.10) we get :

$$Ap^2U^*(s,p) + BpU^*(s,p) = -s^2U^*(s,p) + \frac{I_0\delta\,(e^{-\,p\mu} - e^{-p(\mu+\xi)})}{p\,(s^2+\delta^2)}. \tag{3.11}$$

Hence, $U^*(s,p)$ is given by :

$$U^*(s,p) = \frac{(I_0\,\delta\,C_1)(e^{-p\mu} - e^{-p(\mu+\xi)})}{p(Ap^2+Bp+s^2)(s^2+\delta^2)}. \tag{3.12}$$

The solution, Eq. (3.12) is in the Laplace and Fourier cosine domain and to get the solution in time and space domain we need to take the inverse Laplace and inverse Fourier cosine transforms.

Eq. (3.12) can be written as

$$U^*(s,p) = \frac{I_0\delta\,C_1(e^{-p\mu} - e^{-p(\mu+\xi)})}{(s^2+\delta^2)}\left(\frac{1}{p\,s^2} - \frac{Ap+B}{s^2\,(Bp+Ap^2+s^2)}\right). \tag{3.13}$$

Here, we use the second shifting theorem (SST) in order to the inverse Laplace transform of Eq. (3.13).

Recall that (SST) :

If $L\{\,f(t)\} = F(p)$ then $L\{\,f(t-a)\,u(t-a)\} = e^{-pa}\,F(p)$.

Thus, we have

$$U^*(s,p) = \frac{I_0\delta\,C_1(e^{-p\mu} - e^{-p(\mu+\xi)})}{(s^2+\delta^2)}\left(\frac{1}{p\,s^2} - \frac{Ap+B}{s^2\,(Bp+Ap^2+s^2)}\right)$$

$$= \frac{I_0\delta\,C_1(e^{-p\mu})}{(s^2+\delta^2)}\left(\frac{1}{p\,s^2} - \frac{Ap+B}{s^2\,(Bp+Ap^2+s^2)}\right) - \frac{I_0\delta\,C_1(e^{-p(\mu+\xi)})}{(s^2+\delta^2)}\left(\frac{1}{p\,s^2} - \frac{Ap+B}{s^2\,(Bp+Ap^2+s^2)}\right).$$

Now, let

$$F(p) = \left(\frac{1}{p\, s^2} - \frac{Ap+B}{s^2\,(Bp+Ap^2+s^2)} \right), \text{ then,}$$

$$U^*(s,p) = \frac{I_0 \delta\, C_1}{(s^2+\delta^2)} e^{-p\mu}\, F(p) - \frac{I_0 \delta\, C_1}{(s^2+\delta^2)}\; e^{-p(\mu+\xi)}\, F(p).$$

So that

$$U(s,t) = \frac{I_0 \delta\, C_1}{(s^2+\delta^2)} f(t-\mu)\, u(t-\mu) - \frac{I_0 \delta\, C_1}{(s^2+\delta^2)} f(t-\mu+\xi)\, u(t-\mu+\xi),$$

where,

$$f(t) = L^{-1}\left\{F(p) = \frac{1}{p\, s^2} - \frac{Ap+B}{s^2\,(Bp+Ap^2+s^2)}\right\} \text{ which can be written as,}$$

$$L^{-1}\left\{\frac{1}{p\, s^2}\right\} - L^{-1}\left\{\frac{Ap+B}{s^2\,(Bp+Ap^2+s^2)}\right\}.$$

Now,

$$L^{-1}\left\{\frac{1}{p\, s^2}\right\} = \frac{1}{s^2} \text{ and}$$

$L^{-1}\left\{\frac{Ap+B}{s^2\,(Bp+Ap^2+s^2)}\right\}$ is found by partial fraction decomposition as follows :

$$L^{-1}\left\{\frac{Ap+B}{s^2\,(Bp+Ap^2+s^2)}\right\} = \frac{\left(\begin{array}{c} -Be^{\left(-\frac{B}{2A}-\frac{\sqrt{B^2-4As^2}}{2A}\right)t} + Be^{\left(-\frac{B}{2A}+\frac{\sqrt{B^2-4As^2}}{2A}\right)t} + \\ e^{\left(-\frac{B}{2A}-\frac{\sqrt{B^2-4As^2}}{2A}\right)t}\sqrt{B^2-4As^2} + e^{\left(-\frac{B}{2A}+\frac{\sqrt{B^2-4As^2}}{2A}\right)t}\sqrt{B^2-4As^2} \end{array} \right)}{2s^2\sqrt{B^2-4As^2}}.$$

Thus,

$$f(t) = \frac{1}{s^2} + \frac{\left(\begin{array}{c} -Be^{\left(-\frac{B}{2A}-\frac{\sqrt{B^2-4As^2}}{2A}\right)t} + Be^{\left(-\frac{B}{2A}+\frac{\sqrt{B^2-4As^2}}{2A}\right)t} + \\ e^{\left(-\frac{B}{2A}-\frac{\sqrt{B^2-4As^2}}{2A}\right)t}\sqrt{B^2-4As^2} + e^{\left(-\frac{B}{2A}+\frac{\sqrt{B^2-4As^2}}{2A}\right)t}\sqrt{B^2-4As^2} \end{array} \right)}{2s^2\sqrt{B^2-4As^2}}\, .$$

Therefore $U(s,t)$ becomes, after simplifying and combining the terms

$U(s,t) =$

$$\frac{I_0\, C_1 \delta}{2s^2\sqrt{B^2-4As^2}(s^2+\delta^2)}\left(\begin{array}{c} -e^{-\frac{\left(B+\sqrt{B^2-4As^2}\right)(t-\mu)}{2A}}\left(\begin{array}{c} B\left(-1+e^{\frac{\sqrt{B^2-4As^2}(t-\mu)}{A}}\right)+ \\ \left(\begin{array}{c} 1+e^{\frac{\sqrt{B^2-4As^2}(t-\mu)}{A}} - \\ 2e^{\frac{\left(B+\sqrt{B^2-4As^2}\right)(t-\mu)}{2A}} \end{array}\right)\sqrt{B^2-4As^2} \end{array}\right)\mathrm{u}[t-\mu]+ \\ e^{-\frac{\left(B+\sqrt{B^2-4As^2}\right)(t-\mu-\xi)}{2A}}\left(\begin{array}{c} B\left(-1+e^{\frac{\sqrt{B^2-4As^2}(t-\mu-\xi)}{A}}\right)+ \\ \left(\begin{array}{c} 1+e^{\frac{\sqrt{B^2-4As^2}(t-\mu-\xi)}{A}} - \\ 2e^{\frac{\left(B+\sqrt{B^2-4As^2}\right)(t-\mu-\xi)}{2A}} \end{array}\right)\sqrt{B^2-4As^2} \end{array}\right)\mathrm{u}[t-\mu-\xi] \end{array}\right) \quad (3.14)$$

Now, we get $u(x,t)$ by taking the inverse Fourier cosine transform of Eq. (3.14),

$$u(x,t) = \frac{2}{\pi}\int_0^\infty U(s,t)\cos(sx)\, ds. \quad (3.15)$$

The function $U(s,t)$ has two non-negative singularities at

$s_1 = 0$ and $s_2 = \frac{B}{2\sqrt{A}}$.

The argument used earlier in chapter 2 can be applied here to Eq. (3.14) to deduce that s_1 is a removable singularity and a graphical representation of the integrand shows that the integrand oscillates for $s_2 \geq \frac{B}{2\sqrt{A}}$ and thus the contribution to the integral vanishes due to self-cancellation effect for $s_2 \geq \frac{B}{2\sqrt{A}}$.

We can therefore restrict the range of integration and write:

$$u(x,t) = \frac{2}{\pi}\int_0^{\frac{B}{2\sqrt{A}}} U(s,t)\cos(sx)\, ds.$$

Similarly, for Eq. (3.6) we have $Dv = I_0\delta e^{-\delta x}\, C_2\, p(t)$ which can be solved by the same method above to obtain the following:

so that

$$v(x,t) = \frac{2}{\pi} \int_0^{\frac{B}{2\sqrt{A}}} V(s,t)\cos(sx)\,ds. \tag{3.17}$$

Here $V(s,t)$ is same as $U(s,t)$ except that we replace C_1 by C_2 .

Therefore, the final solution of this problem is given by Eq. (3.7) which is

$$T(x,t) = u(x,t) - v(x,t)$$

$$= \frac{2}{\pi} \int_0^{\frac{B}{2\sqrt{A}}} U(s,t)\cos(sx)\,ds - \frac{2}{\pi} \int_0^{\frac{B}{2\sqrt{A}}} V(s,t)\cos(sx)\,ds, \tag{3.18}$$

where, $U(s,t)$ and $V(s,t)$ are given by Eq. (3.14) and Eq. (3.16) respectively.

These integrals have been evaluated numerically using Mathematica to obtain the value of $T(x,t)$ at given (x,t). These results are given in Fig. 3.1. – Fig. 3.4.

3.2.3 Thermal-Stress Study

In this part, we study the thermal stress influence on this heating process due to the step input laser pulse volumetric heat source.

As in chapter 2, The equation of the thermal stress is given by the following PDE [21] :

$$\sigma_{xx} - m^2\sigma_{tt} = n\,T_{tt}, \qquad (3.19)$$

where,

$m = \sqrt{\frac{(1+v)(1-2\,v)\rho}{E\,(1-v)}}$ and

$n = \frac{(1+v)}{(1-v)}\,\rho\,\alpha_T$.

The initial and boundary conditions considered here are :

$$\sigma(x,0) = 0,$$

$$\frac{\partial}{\partial t}\sigma(x,0) = 0. \qquad (3.20)$$

In Eq. (3.20), initially, it is assumed that thermal stress is zero inside the substrate material and time derivative of stress is also zero.

$$\lim_{x\to\infty}\sigma(x,t) = 0,$$

$$\frac{\partial}{\partial x}\sigma(0,t) = 0. \qquad (3.21)$$

Equation (3.21) expresses that the stress at large distance (depth) is zero and that surface stress gradient also vanishes.

Now, on taking the Fourier transform of Eq. (3.19) we get:

$$-s^2\bar{\sigma}(s,t) - \sigma_x(0,t) - h^2\frac{d^2}{dt^2}\bar{\sigma}(s,t) = q\,\frac{d^2}{dt^2}\bar{T}(s,t). \qquad (3.22)$$

Applying the transformed boundary conditions for Eq. (3.22) we obtain

$$-s^2\bar{\sigma}(s,t) - m^2\frac{d^2}{dt^2}\bar{\sigma}(s,t) = n\,\frac{d^2}{dt^2}\bar{T}(s,t). \tag{3.23}$$

Eq. (3.23) can be solved by applying the Laplace transform with respect to t.

Applying the Laplace transform to Eq. (3.23) yields:

$$-s^2\bar{\sigma}^*(s,p) - m^2p^2\,\bar{\sigma}^*(s,p) - p\,\bar{\sigma}(s,0) - \bar{\sigma}_t(s,0) = n\,p^2\bar{T}^*(s,p). \tag{3.24}$$

Now, applying the transformed initial conditions to Eq. (3.24) we get:

$$-s^2\bar{\sigma}^*(s,p) - m^2p^2\,\bar{\sigma}^*(s,p) = n\,p^2\bar{T}^*(s,p). \tag{3.25}$$

Hence, we obtain:

$$\bar{\sigma}^*(s,p) = \frac{-n\,p^2}{(m^2p^2+s^2)}\,\bar{T}^*(s,p). \tag{3.26}$$

The solution, Eq. (3.26), is in the transformed domain and to get the solution in time and space domain we need to take the inverse Laplace and inverse Fourier cosine transforms.

Taking the Laplace and Fourier cosine transforms of Eq. (3.7) gives:

$$\bar{T}^*(s,p) = U^*(s,p) - V^*(s,p)\,. \tag{3.27}$$

Substitute Eq. (3.27) into Eq. (3.26) we get:

$$\bar{\sigma}^*(s,p) = \frac{-n\,p^2}{(m^2p^2+s^2)}[U^*(s,p) - V^*(s,p)\,]. \tag{3.28}$$

Now, let

$$\overline{\emptyset_1}^*(s,p) = \frac{-n\,p^2}{(m^2p^2+s^2)}\,U^*(s,p) \tag{3.29}$$

and

$$\overline{\emptyset_2}^*(s,p) = \frac{-n\,p^2}{(m^2p^2+s^2)}\,V^*(s,p), \tag{3.30}$$

where, U^* and V^* were the functions found earlier in the previous part.

Now, $\overline{\emptyset_1}^*(s,p)$ becomes :

$$\overline{\emptyset_1}^*(s,p) = \frac{-n\,p^2}{(m^2p^2+s^2)}\,\frac{(I_0\,\delta\,C_1)\left(e^{-p\mu}-e^{-p(\mu+\xi)}\right)}{p(Ap^2+Bp+s^2)(s^2+\delta^2)}. \tag{3.31}$$

Similarly for $\overline{\emptyset_2}^*(s,p)$.

Then, we obtain the thermal stress using:

$$\sigma(x,t) = \emptyset_1(x,t) - \emptyset_2(x,t). \tag{3.32}$$

Now, we take Laplace inverse transform of $\overline{\emptyset_1}^*(s,p)$ as follows:

$$\overline{\emptyset_1}^*(s,p) = \frac{-n\,p^2}{(m^2p^2+s^2)}\,\frac{(I_0\,\delta\,C_1)\left(e^{-p\mu}-e^{-p(\mu+\xi)}\right)}{p(Ap^2+Bp+s^2)(s^2+\delta^2)}$$

$$= \frac{-(I_0\,\delta\,n\,C_1)\left(e^{-p\mu}-e^{-p(\mu+\xi)}\right)}{(s^2+\delta^2)}\left(\frac{p}{(m^2p^2+s^2)(Ap^2+Bp+s^2)}\right).$$

Now, use Second shifting theorem (SST) as in the previous part by letting:

$$F(p) = \frac{p}{(m^2p^2+s^2)(Ap^2+Bp+s^2)}.$$

$f(t)$ then is found by partial fraction decomposition.

We find and simplify to get:

$f(t) =$

$$\frac{\left(\begin{array}{c} ABe^{\left(-\frac{B}{2A}-\frac{\sqrt{B^2-4As^2}}{2A}\right)t} - ABe^{\left(-\frac{B}{2A}+\frac{\sqrt{B^2-4As^2}}{2A}\right)t} + \\ Be^{\left(-\frac{B}{2A}-\frac{\sqrt{B^2-4As^2}}{2A}\right)t} m^2 - Be^{\left(-\frac{B}{2A}+\frac{\sqrt{B^2-4As^2}}{2A}\right)t} m^2 + \\ Ae^{\left(-\frac{B}{2A}-\frac{\sqrt{B^2-4As^2}}{2A}\right)t} \sqrt{B^2-4As^2} + \\ Ae^{\left(-\frac{B}{2A}+\frac{\sqrt{B^2-4As^2}}{2A}\right)t} \sqrt{B^2-4As^2} - \\ e^{\left(-\frac{B}{2A}-\frac{\sqrt{B^2-4As^2}}{2A}\right)t} m^2\sqrt{B^2-4As^2} - \\ e^{\left(-\frac{B}{2A}+\frac{\sqrt{B^2-4As^2}}{2A}\right)t} m^2\sqrt{B^2-4As^2} \end{array}\right)}{2\sqrt{B^2-4As^2}(B^2m^2+A^2s^2-2Am^2s^2+m^4s^2)} + \frac{-As\mathrm{Cos}\left[\frac{st}{m}\right]+m^2s\mathrm{Cos}\left[\frac{st}{m}\right]+Bm\mathrm{Sin}\left[\frac{st}{m}\right]}{s(B^2m^2+A^2s^2-2Am^2s^2+m^4s^2)}.$$

Therefore $\overline{\emptyset_1}(s,t)$ becomes, after simplifying and combining the terms

$\overline{\emptyset_1}(s,t) =$

$$\frac{-(I_0\ \delta\ n\ C_1)}{(s^2+\delta^2)} \times$$

$$\left(\frac{\left(e^{-\frac{\left(B+\sqrt{B^2-4As^2}\right)(t-\mu)}{2A}}\left(A\left(\begin{array}{c} B - Be^{\frac{\sqrt{B^2-4As^2}(t-\mu)}{A}} + \\ \left(1+e^{\frac{\sqrt{B^2-4As^2}(t-\mu)}{A}}\right)\sqrt{B^2-4As^2}\end{array}\right) - m^2\left(\begin{array}{c} B\left(-1+e^{\frac{\sqrt{B^2-4As^2}(t-\mu)}{A}}\right) + \\ \left(1+e^{\frac{\sqrt{B^2-4As^2}(t-\mu)}{A}}\right)\sqrt{B^2-4As^2}\end{array}\right)\right)\right)}{\left(2\sqrt{B^2-4As^2}(B^2m^2+A^2s^2-2Am^2s^2+m^4s^2)\right)} + \frac{-As\mathrm{Cos}\left[\frac{s(t-\mu)}{m}\right]+m^2s\mathrm{Cos}\left[\frac{s(t-\mu)}{m}\right]+Bm\mathrm{Sin}\left[\frac{s(t-\mu)}{m}\right]}{s(B^2m^2+A^2s^2-2Am^2s^2+m^4s^2)}\right)\mathrm{u}[t-\mu] -$$

$$\frac{-(I_0\,\delta\,n\,C_1)}{(s^2+\delta^2)} \times$$

$$\left(\frac{\left(e^{-\frac{\left(B+\sqrt{B^2-4As^2}\right)(t-\mu-\xi)}{2A}}\left(A\left(B-Be^{\frac{\sqrt{B^2-4As^2}(t-\mu-\xi)}{A}}+\left(1+e^{\frac{\sqrt{B^2-4As^2}(t-\mu-\xi)}{A}}\right)\sqrt{B^2-4As^2}\right)-m^2\left(B\left(-1+e^{\frac{\sqrt{B^2-4As^2}(t-\mu-\xi)}{A}}\right)+\left(1+e^{\frac{\sqrt{B^2-4As^2}(t-\mu-\xi)}{A}}\right)\sqrt{B^2-4As^2}\right)\right)\right)}{\left(2\sqrt{B^2-4As^2}(B^2m^2+A^2s^2-2Am^2s^2+m^4s^2)\right)} + \frac{-As\mathrm{Cos}\left[\frac{s(t-\mu-\xi)}{m}\right]+m^2s\mathrm{Cos}\left[\frac{s(t-\mu-\xi)}{m}\right]+Bm\mathrm{Sin}\left[\frac{s(t-\mu-\xi)}{m}\right]}{s(B^2m^2+A^2s^2-2Am^2s^2+m^4s^2)}\right)\mathrm{u}[t-\mu-\xi]\,. \qquad (3.33)$$

$\overline{\emptyset_2}(s,t)$ can be found by replacing C_1 with C_2 in Eq. (3.33).

Now, we find $\emptyset_1(x,t)$ by taking the inverse Fourier cosine transform of $\overline{\emptyset_1}(s,t)$,

$$\emptyset_1(x,t) = \frac{2}{\pi}\int_0^{\infty}\overline{\emptyset_1}(s,t)\cos(sx)\,ds. \qquad (3.34)$$

$\overline{\emptyset_1}(s,t)$ has two non-negative singularities at

$sg_1 = 0$, and $sg_2 = \frac{B}{2\sqrt{A}}$.

An argument similar to that in Eq. (3.15), reveals that sg_1 is a removable singularity. Also, we reduce the range of integration and write:

$$\emptyset_1(x,t) = \frac{2}{\pi}\int_0^{\frac{B}{2\sqrt{A}}}\overline{\emptyset_1}(s,t)\cos(sx)\,ds. \qquad (3.35)$$

Similarly,

$$\emptyset_2(x,t) = \frac{2}{\pi}\int_0^{\frac{B}{2\sqrt{A}}}\overline{\emptyset_2}(s,t)\cos(sx)\,ds. \qquad (3.36)$$

So Eq. (3.32) becomes

$$\sigma(x,t) = \frac{2}{\pi}\int_0^{\frac{B}{2\sqrt{A}}} \overline{\emptyset_1}(s,t)\cos(sx)\,ds - \frac{2}{\pi}\int_0^{\frac{B}{2\sqrt{A}}} \overline{\emptyset_2}(s,t)\cos(sx)\,ds. \tag{3.37}$$

These integrals were evaluated numerically using Mathematica to obtain $\sigma(x,t)$. The results are given in Fig. 3.5 – Fig. 3.8 .

3.2.4 Results and discussion

Analytical solution for the hyperbolic heat conduction equation accounting for finite speed of heat conduction with presence of the volumetric heat source is presented. The step input short pulse laser source is incorporated as a volumetric source in the equation. We have used the Laplace transform in time and the Fourier cosine transform in space to find the solution of the hyperbolic heat conduction equation incorporating the appropriate initial and boundary conditions. The inversion of the Laplace transform is analytically performed. For inversion of the Fourier cosine transform, Mathematica is used and solution displayed graphically. Furthermore, thermal stress field is obtained through coupling heat and thermal stress equations and the integral transforms are used to obtain the solutions of the coupled equations. The inversion of the Laplace transform is performed using second shifting theorem of Laplace transform while Mathematica is used for the inverse Fourier cosine transform. Also, singularities of integrand in the transformed domains have investigated and found that all are removed and avoided through reducing the range of integration, which can be justified due to the oscillatory nature of the integral because of self-cancellation effect.

3.2.5 Tables

Table 3.1: The properties and the range of values used in the simulations.

α ($\mathrm{m^2}/\mathrm{s}$)	$\mathbf{0.227 \times 10^{-4}}$
C_1 ($1/\mathrm{s}$)	2×10^{-2}
δ ($1/\mathrm{m}$)	6.16×10^{7}
I_0 ($\mathrm{W}/\mathrm{m^2}$)	4×10^{20}
C_2 ($1/\mathrm{s}$)	6 and 14×10^{-2}
A ($\mathrm{s^2}/\mathrm{m^2}$)	2.1×10^{-10}
B ($\mathrm{s}/\mathrm{m^2}$)	44052.86
V	0.3
E ($1/\mathrm{K}$)	200×10^{9}
ρ ($\mathrm{kg}/\mathrm{m^3}$)	7850
α_T ($1/\mathrm{s}$)	12×10^{-6}

3.2.6 Figures

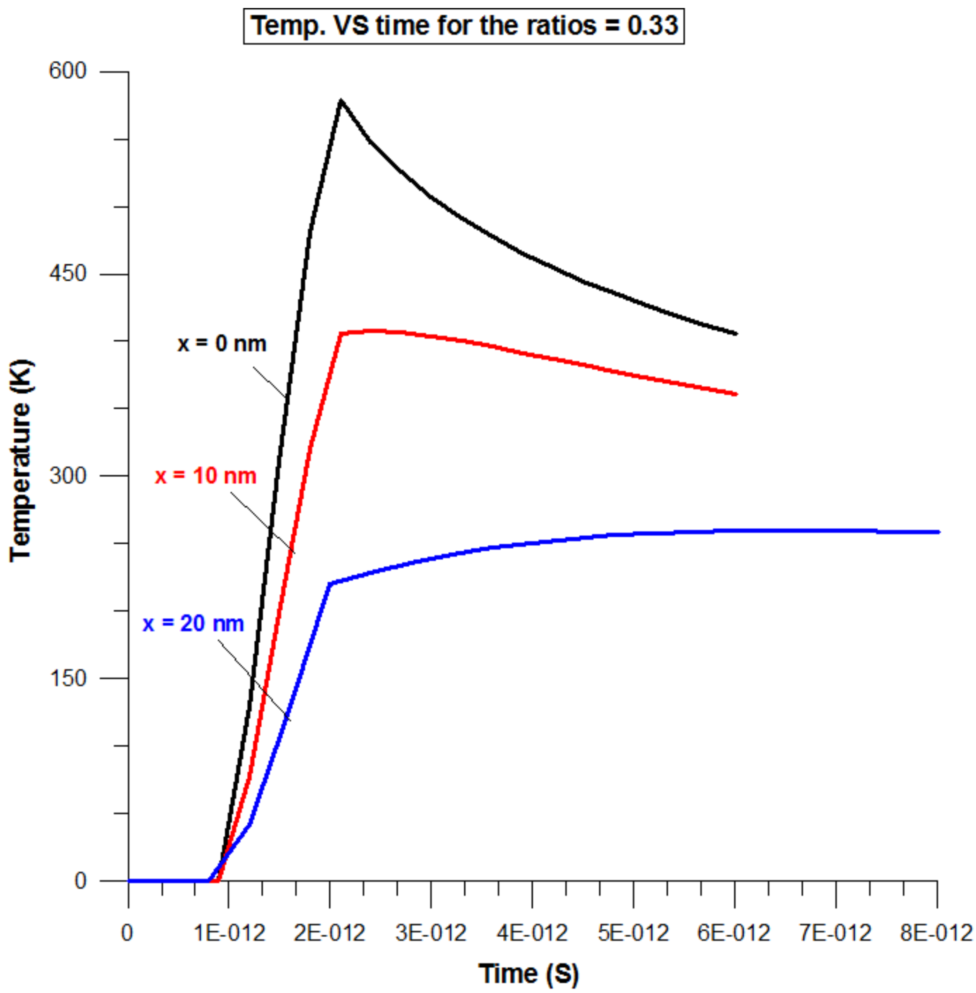

Figure 3.1: Temperature Variation with time at different depths for the pulse ratio = 0.33

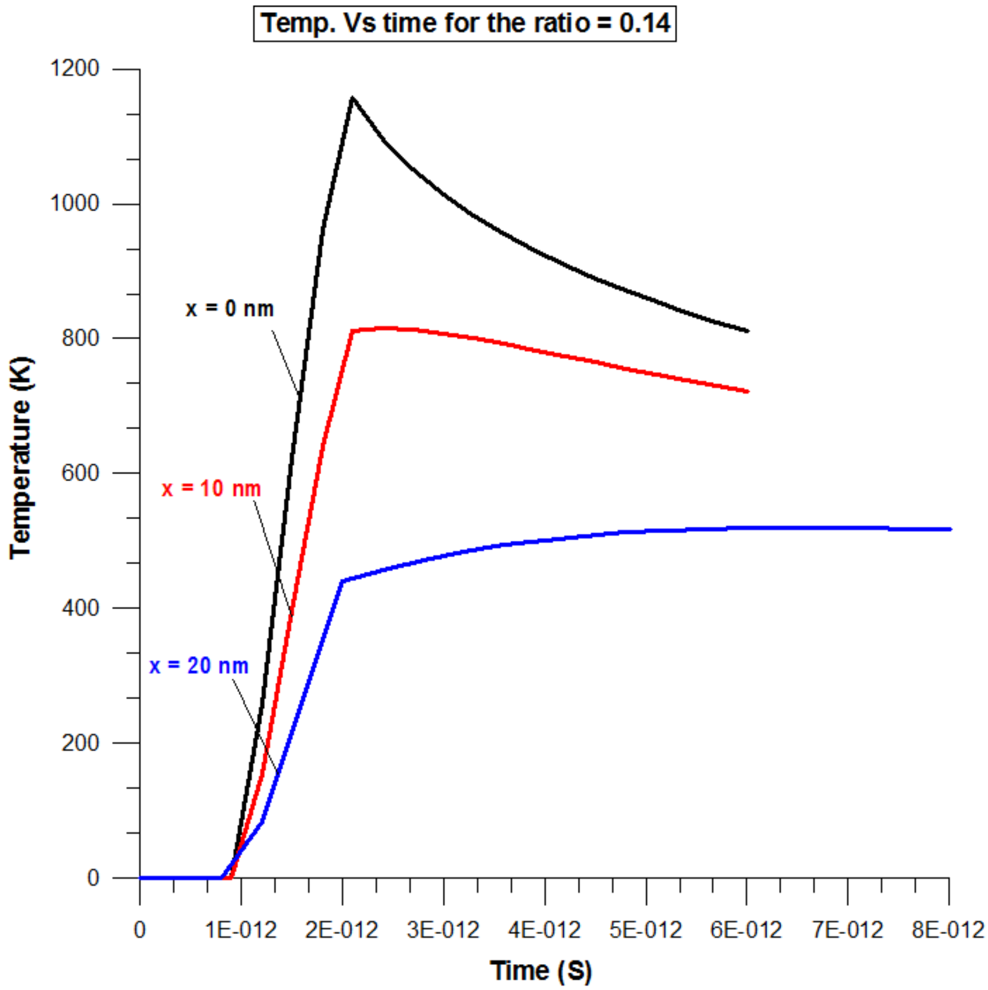

Figure 3.2: Temperature Variation with time at different depths for the pulse ratio = 0.14

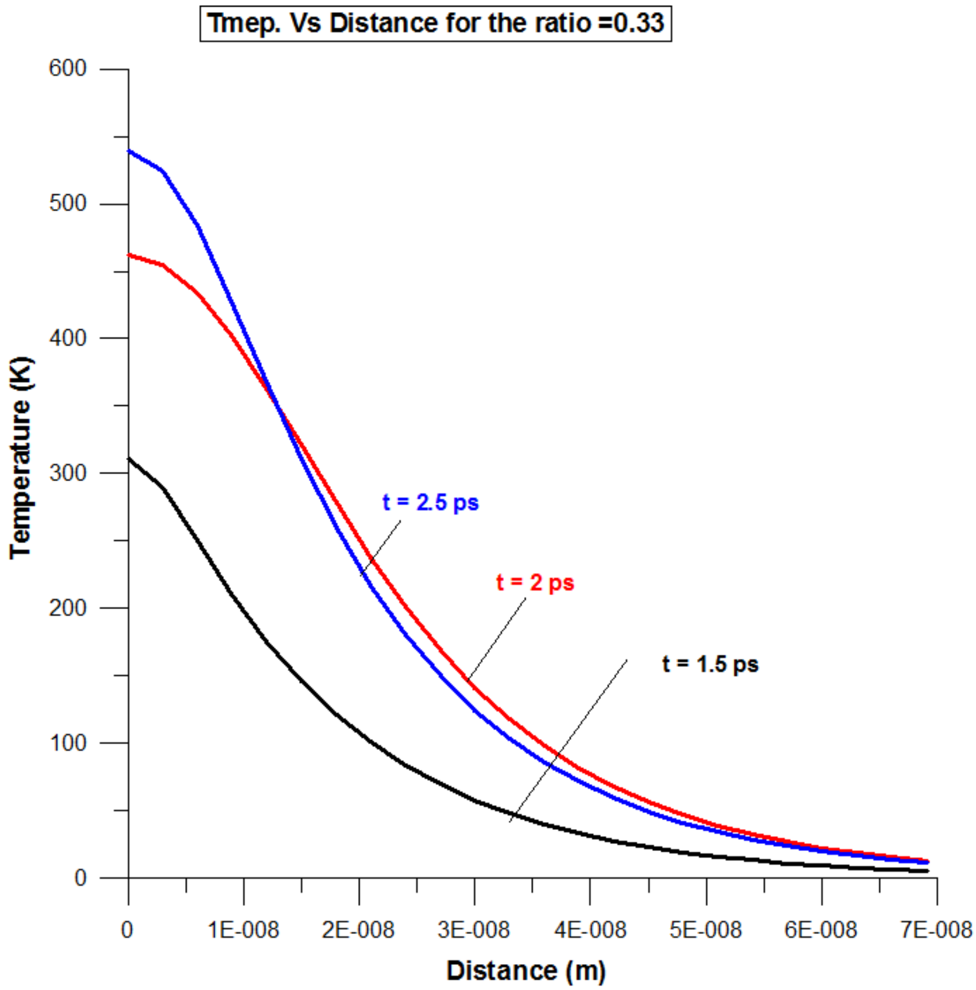

Figure 3.3: Temperature Variation inside the substrate material for different heating periods for the pulse ratio = 0.33

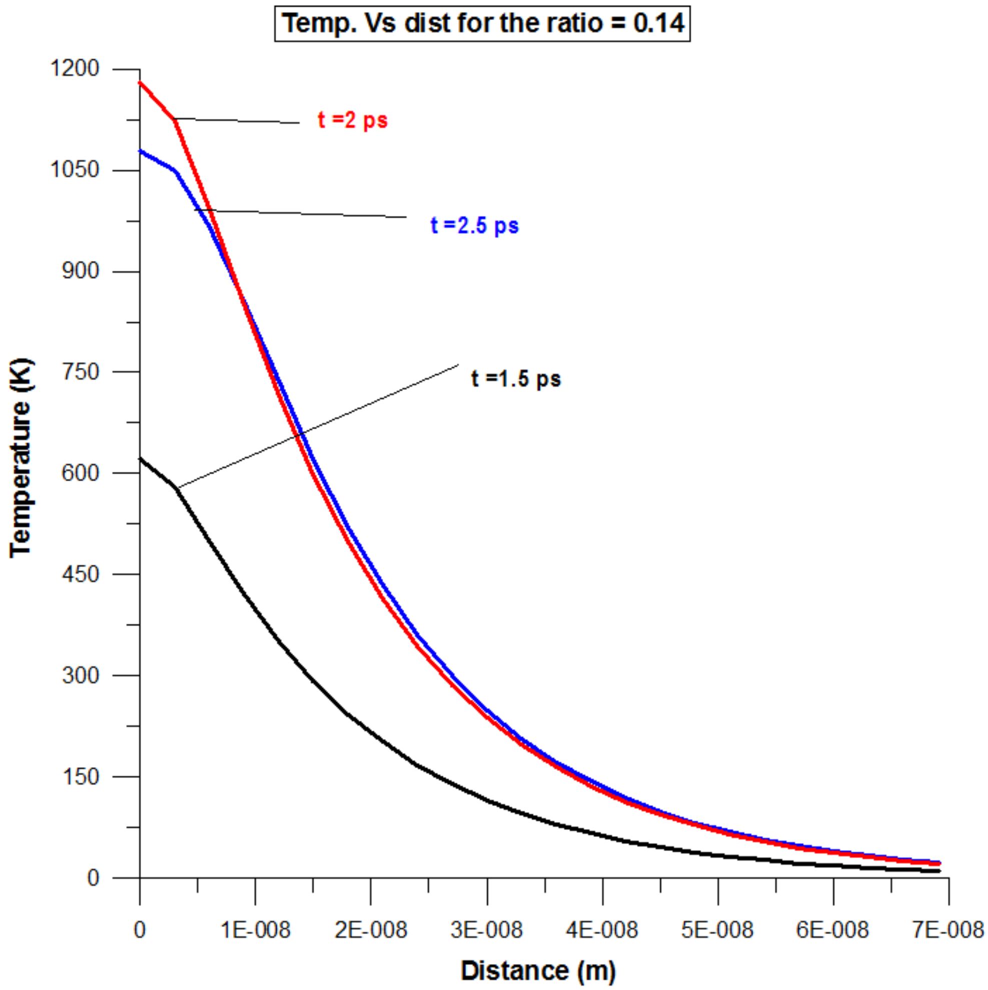

Figure 3.4: Temperature Variation inside the substrate material for different heating periods for the pulse ratio = 0.14

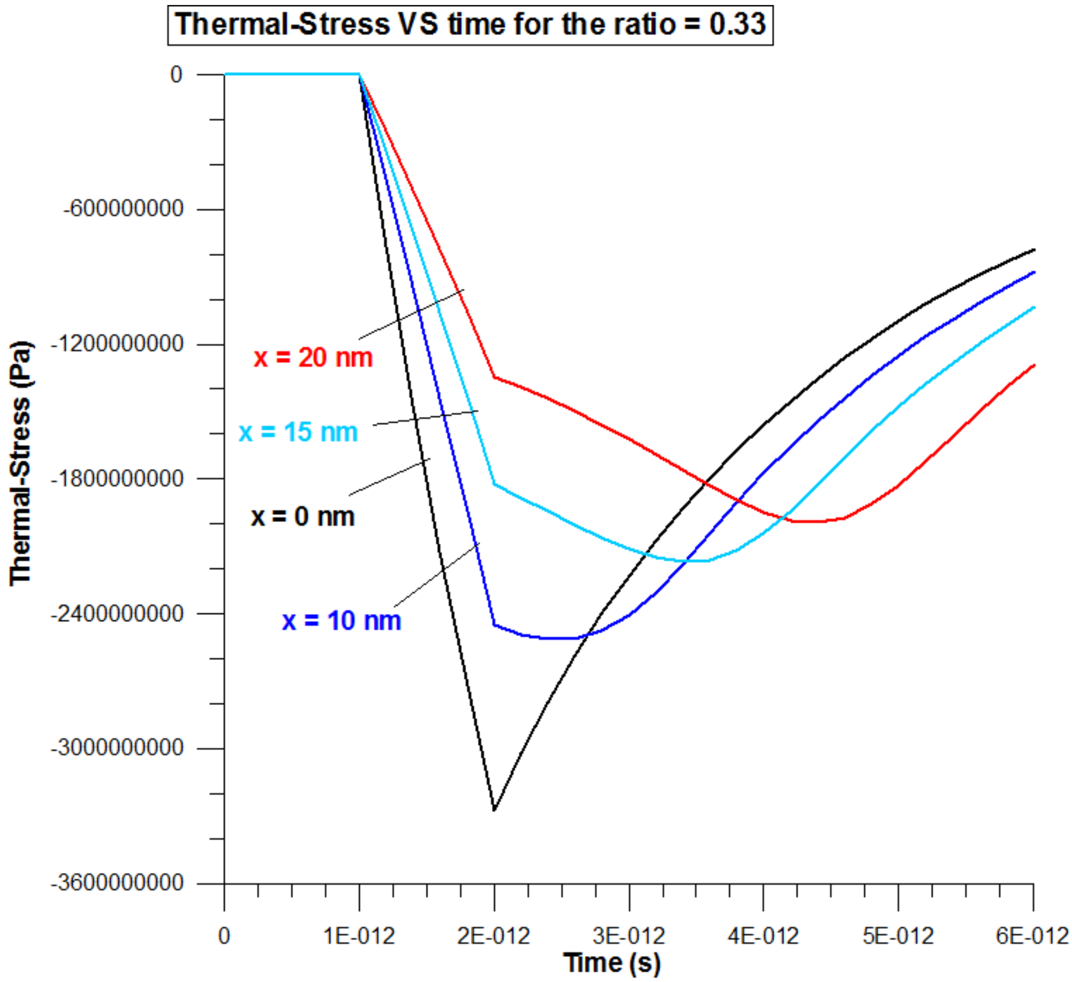

Figure 3.5: Thermal-Stress Variation with time at different depths for the pulse ratio = 0.33

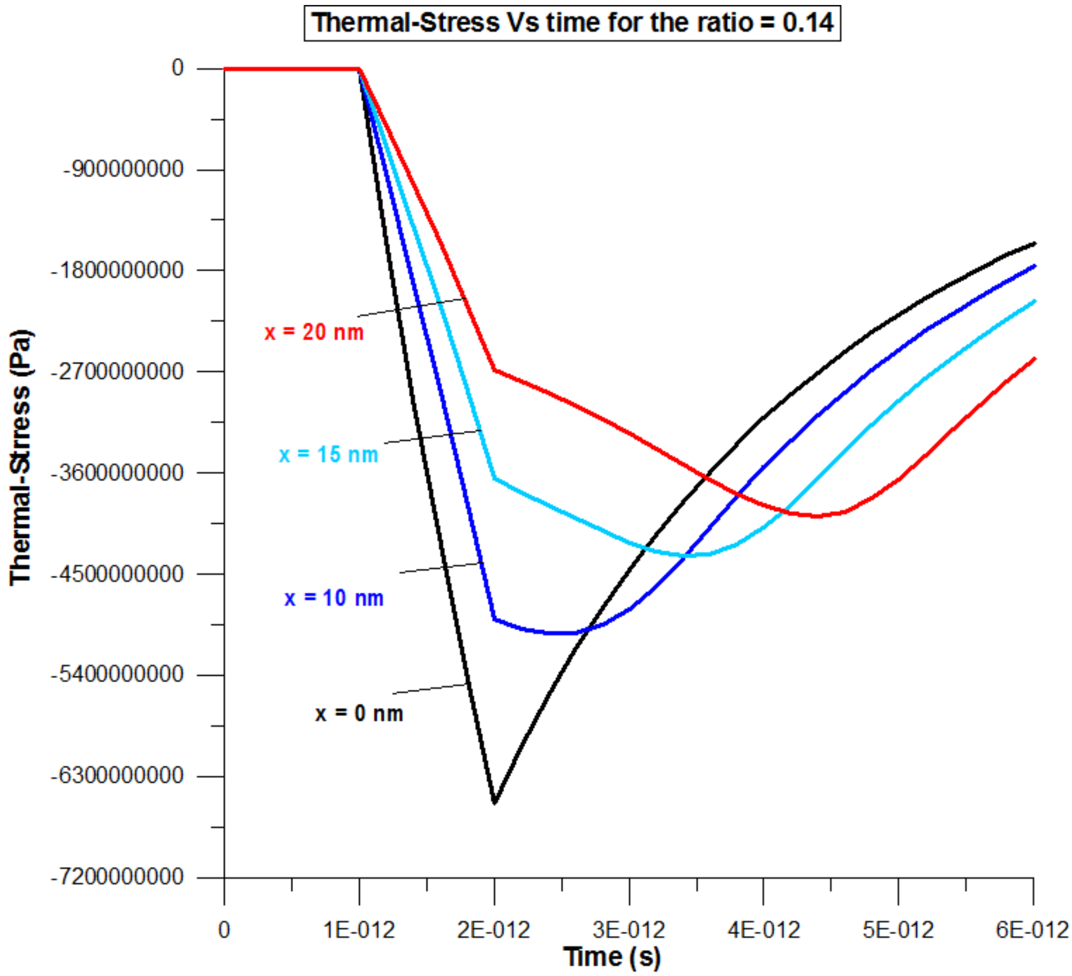

Figure 3.6: Thermal-Stress Variation with time at different depths for the pulse ratio = 0.14

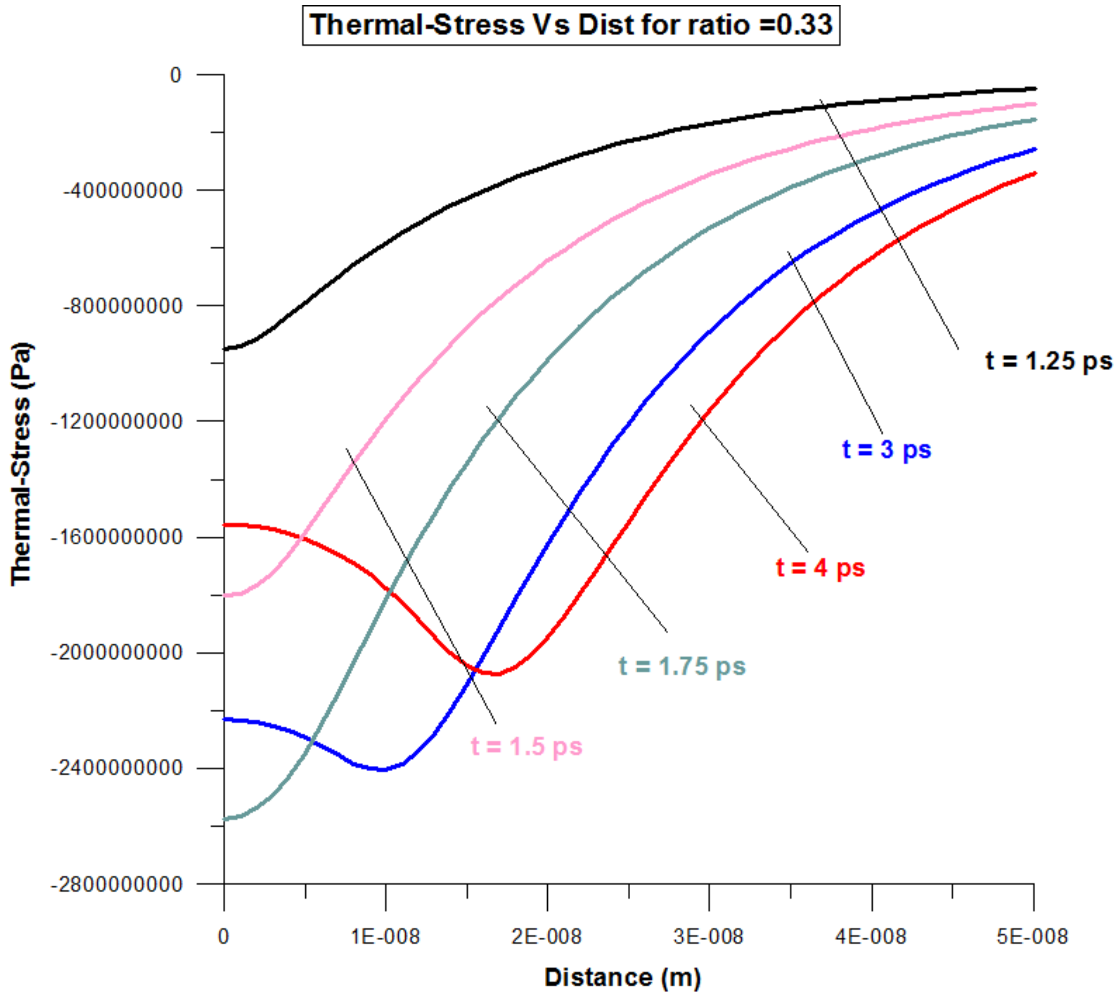

Figure 3.7: Thermal-Stress Variation inside the substrate material for different heating periods for the pulse ratio = 0.33

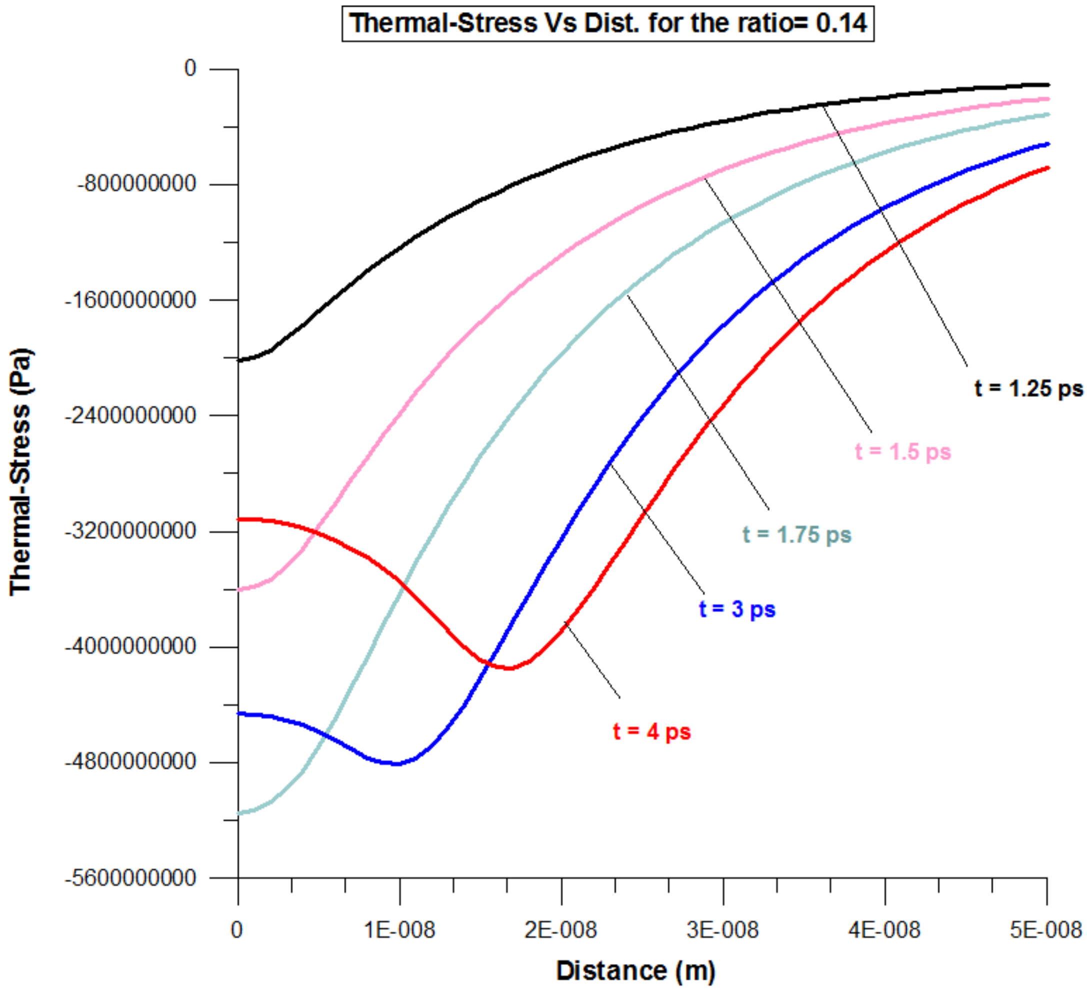

Figure 3.8: Thermal-Stress Variation inside the substrate material for different heating periods for the pulse ratio = 0.33

Chapter 4

Thermal Stress in a Half Space with Mixed Boundary Conditions due to Time Dependent Heat Source.

4.1 Introduction

In this chapter, we consider mixed boundary value problem in thermal stress in half space. The surface of the half space is heated by a time dependent source which produces temperature changes in the material. The resulting thermal stresses are our main aim in this study. We assume that the surface of the half space satisfies mixed boundary conditions. In the part of the boundary ($x < 0$) is stress free while in the remaining boundary ($x > 0$), the gradient of the stress vanishes. The heat conduction equation is solved using the Laplace transform in time and the Fourier transform in space variable. To deal with the determination of thermal stress we employ the Jones modification of the so-called Wiener-Hopf technique [12].

However, we have studied two types of heat sources. In the first case, a time exponentially heat source while in the second case an instantaneous line source has been considered.

4.2 Thermal stress in a half space with mixed boundary conditions due to a time exponentially heat source.

4.2.1 Formulation of the problem

We consider two dimensional heat conduction equation in half space ($y > 0$), which is governing by the following PDE :

$$\phi_{xx}(x,y,t) + \phi_{yy}(x,y,t) = \frac{1}{k}\,\phi_t(x,y,t) + f(x,t), \tag{4.1}$$

where,

$f(x,t) = I_0\,\delta(x-a)\,e^{-\beta t}$ is the heat source ,

$\delta(x-a)$ is the Dirac delta function. As for, β , a are the source parameters and I_0 is a positive coefficient.

The initial condition are given by:

$$\phi(x,y,0) = 0 \ , \begin{cases} -\infty < x < \infty \\ 0 \le y < \infty \end{cases} \tag{4.2}$$

while the boundary conditions are:

$$\lim_{y\to\infty} \phi(x,y,t) = 0 \quad , \quad \begin{cases} -\infty < x < \infty \\ t \ge 0 \end{cases}$$

$$\phi_y(x,0,t) = 0 \quad , \quad \begin{cases} -\infty < x < \infty \\ t \ge 0 \end{cases} \tag{4.3}$$

The stress coupled equation is given by:

$$\sigma_{xx}(x,y,t) + \sigma_{yy}(x,y,t) - \mu^2\sigma_{tt}(x,y,t) = \lambda\,\phi_{tt}(x,y,t), \tag{4.4}$$

here,

μ and λ are positive coefficients.

ϕ_{tt} is the second derivative resulting from the solution of Eq. (4.1).

The initial conditions are assumed as follows:

$$\sigma(x,y,0)=0, \qquad \begin{cases} -\infty < x < \infty \\ 0 \le y < \infty \end{cases}$$

$$\sigma_t(x,y,0)=0\ , \qquad \begin{cases} -\infty < x < \infty \\ 0 \le y < \infty \end{cases} \tag{4.5}$$

while the boundary conditions are:

$$\sigma(x,0,t)=0\ , \qquad \begin{cases} -\infty < x < 0 \\ t \ge 0 \end{cases}$$

$$\sigma_y(x,0,t)=0, \quad \begin{cases} 0 < x < \infty \\ t \ge 0 \end{cases} \tag{4.6}$$

Also,

$$\lim_{y\to\infty}\sigma(x,y,t)=0. \qquad \begin{cases} -\infty < x < \infty \\ t \ge 0 \end{cases} \tag{4.7}$$

4.2.2 The Wiener-Hopf Equation

The Laplace transform in the time variable t and its inverse transform in s are defined by:

$$L\{\sigma(x,y,t)\}=\int_0^\infty \sigma(x,y,t)e^{-st}dt=\overline{\sigma}(x,y,s) \tag{4.8}$$

and

$$L^{-1}\{\overline{\sigma}(x,y,s)\}=\frac{1}{2\pi i}\int_{i\infty-c}^{i\infty+c}\overline{\sigma}(x,y,s)\,e^{st}ds=\sigma(x,y,t). \tag{4.9}$$

In the same way, we define the Fourier transform in x and its corresponding inverse Fourier transform in α by:

$$\mathcal{F}\{\sigma(x,y,t)\}=\int_{-\infty}^{\infty}\sigma(x,y,t)e^{i\alpha x}dx=\sigma^*(\alpha,y,t) \tag{4.10}$$

and

$$\mathcal{F}^{-1}\{\sigma^*(\alpha,y,t)\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sigma^*(\alpha,y,t)e^{-i\alpha x}d\alpha=\sigma(x,y,t), \tag{4.11}$$

with $\alpha = \xi + i\tau$.

Moreover, we also introduce the half range Fourier transforms as

$$\int_0^\infty \sigma(x,y,t)\, e^{i\alpha x} dx = \sigma_+^*(\alpha,y,t) \tag{4.12}$$

and

$$\int_{-\infty}^0 \sigma(x,y,t) e^{i\alpha x} dx = \sigma_-^*(\alpha,y,t). \tag{4.13}$$

So that,

$$\sigma^*(\alpha,y,t) = \sigma_+^*(\alpha,y,t) + \sigma_-^*(\alpha,y,t). \tag{4.14}$$

Where

$\sigma_+^*(\alpha,y,t) = O(e^{\tau_- x})$ as $x \to \infty$ and $\sigma_-^*(\alpha,y,t) = 0(e^{\tau_+ x})$ as $x \to -\infty$, See [12].

Thus $\sigma_+^*(\alpha,y,t)$ is an analytic function of α in the upper half-plane $\tau > \tau_-$, while $\sigma_-^*(\alpha,y,t)$ is an analytic function of α in the lower half-plane $\tau < \tau_+$ respectively. Therefore, $\sigma^*(\alpha,y,t)$ defined an analytic function in the common strip $\tau_- < \tau < \tau_+$ with $\tau = Im(\alpha)$.

4.2.3 Solution of the heat equation

Taking Laplace transform in t of Eq. (4.1) we get:

$$\bar{\phi}_{xx}(x,y,s) + \bar{\phi}_{yy}(x,y,s) = \frac{s}{k}\bar{\phi}(x,y,t) - \frac{1}{k}\phi(x,y,0) + \bar{f}(x,s). \tag{4.15}$$

Applying the initial condition to get:

$$\bar{\phi}_{xx}(x,y,s) + \bar{\phi}_{yy}(x,y,s) = \frac{s}{k}\bar{\phi}(x,y,t) + \bar{f}(x,s). \tag{4.16}$$

Taking Fourier transform in x of Eq. (4.16) we obtain :

$$-\alpha^2\bar{\phi}^*(\alpha,y,s) + \bar{\phi}^*{}_{yy}(\alpha,y,s) = \frac{s}{k}\bar{\phi}^*(\alpha,y,t) + \bar{f}^*(\alpha,s), \tag{4.17}$$

which can be written as,

$$\bar{\phi}^*{}_{yy}(\alpha,y,s) - \left(\alpha^2 + \frac{s}{k}\right)\bar{\phi}^*(\alpha,y,s) = \bar{f}^*(\alpha,s). \tag{4.18}$$

Therefore the solution is,

$$\bar{\phi}^*(\alpha, y, s) = A\, e^{-\sqrt{\alpha^2+\frac{s}{k}}\, y} + B\, e^{\sqrt{\alpha^2+\frac{s}{k}}\, y} - \frac{\bar{f}^*(\alpha,s)}{\left(\alpha^2+\frac{s}{k}\right)}, \tag{4.19}$$

where,

$$\bar{f}^*(\alpha, s) = \frac{I_0\, e^{i\, a\, \alpha}}{(s+\beta)}\, .$$

Now, using boundary conditions in Eq. (4.3) we deduce that :

$B = 0$ and $A = 0$

Thus, Eq. (4.19) becomes :

$$\bar{\phi}^*(\alpha, y, s) = -\, \frac{I_0\, e^{i\, a\, \alpha}}{(s+\beta)\left(\alpha^2+\frac{s}{k}\right)}. \tag{4.20}$$

By Partial fraction decomposition, the inverse Laplace transform of Eq. (4.20) is obtained:

$$\phi^*(\alpha, y, t) = -\, \frac{k\, I_0\, e^{i\, a\, \alpha}\left(e^{-kt\alpha^2} - e^{-t\beta}\right)}{k\alpha^2 - \beta}\, . \tag{4.21}$$

So that $\phi(x, y, t)$ is given by,

$$\phi(x, y, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi^*(\alpha, y, t) e^{-i\, \alpha\, x}\; d\alpha. \tag{4.22}$$

4.2.4 Solution of the stress equation

Taking Fourier transform in x of Eq. (4.4) we obtain :

$$-\, \alpha^2 \sigma^*(\alpha, y, t) + \sigma^*{}_{yy}(\alpha, y, t) - \mu^2 \sigma^*{}_{tt}(\alpha, y, t) = \lambda\, \phi^*{}_{tt}(\alpha, y, t). \tag{4.23}$$

Now, taking Laplace transform in t we get:

$$-\, \alpha^2 \overline{\sigma}^*(\alpha, y, s) + \overline{\sigma}^*{}_{yy}(\alpha, y, s) - \mu^2 s^2 \overline{\sigma}^*(\alpha, y, s) - \mu^2 s\, \sigma^*(\alpha, y, 0) - \mu^2 \sigma^*{}_t(\alpha, y, 0) =$$
$$\lambda\, s^2 \bar{\phi}^*(\alpha, y, s) - \lambda\, \bar{\phi}(\alpha, y, 0) - \lambda\, \overline{\phi_t}(\alpha, y, 0). \tag{4.24}$$

Applying the transformed initial conditions we get:

$$-\, \alpha^2 \overline{\sigma}^*(\alpha, y, s) + \overline{\sigma}^*{}_{yy}(\alpha, y, s) - \mu^2 s^2 \overline{\sigma}^*(\alpha, y, s) = \lambda\, s^2 \bar{\phi}^*(\alpha, y, t) - \lambda\, \overline{\phi_t}(\alpha, y, 0). \tag{4.25}$$

$\overline{\phi_t}(\alpha, y, 0)$ is computed using Eq. (4.21) to get :

$\overline{\phi_t}(\alpha, y, 0) = -k\, I_0\, e^{i\,a\,\alpha},$

so that Eq. (4.25) becomes,

$$\overline{\sigma}^*{}_{yy}(\alpha, y, t) - (\alpha^2 + \mu^2 s^2)\, \overline{\sigma}^*(\alpha, y, t) = -\frac{\lambda\, s^2\, I_0\, e^{i\,a\,\alpha}}{(s+\beta)\left(\alpha^2+\frac{s}{k}\right)} + \lambda\, k\, I_0\, e^{i\,a\,\alpha}, \tag{4.26}$$

Therefore, the solution is,

$$\overline{\sigma}^*(\alpha, y, s) = \begin{pmatrix} C(\alpha) e^{-\sqrt{\alpha^2+\mu^2 s^2}\; y} + D(\alpha)\, e^{\sqrt{\alpha^2+\mu^2 s^2}\; y} + \\ \frac{\lambda\, s^2\, I_0\, e^{i\,a\,\alpha}}{(s+\beta)\,(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} - \frac{\lambda\, k\, I_0\, e^{i\,a\,\alpha}}{(\alpha^2+\mu^2 s^2)} \end{pmatrix}. \tag{4.27}$$

Using the boundary in Eq. (4.7) we deduce that $D(\alpha) = 0$.

hence,

$$\overline{\sigma}^*(\alpha, y, s) = C(\alpha)\, e^{-\sqrt{\alpha^2+\mu^2 s^2}\; y} + \frac{\lambda\, s^2\, I_0\, e^{i\,a\,\alpha}}{(s+\beta)\,(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} - \frac{\lambda\, k\, I_0\, e^{i\,a\,\alpha}}{(\alpha^2+\mu^2 s^2)}. \tag{4.28}$$

We, now transform the boundaries in Eq. (4.6) as follows:

$$\sigma(x, 0, t) = 0, \quad \begin{cases} -\infty < x < 0 \\ t \geq 0 \end{cases} \Rightarrow \overline{\sigma}^*{}_-(\alpha, 0, s) = 0 \tag{4.29}$$

$$\sigma_y(x, 0, t) = 0, \quad \begin{cases} 0 < x < \infty \\ t \geq 0 \end{cases} \Rightarrow \overline{\sigma_-}^{*\prime}(\alpha, 0, s) = 0 \tag{4.30}$$

where $\{\,'\,\}$ denotes the derivative of $\overline{\sigma}^*$ with respect to y.

so that,

$$\overline{\sigma}^*{}_+(\alpha, 0, s) = C(\alpha) + \frac{\lambda\, s^2\, I_0\, e^{i\,a\,\alpha}}{(s+\beta)\,(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} - \frac{\lambda\, k\, I_0\, e^{i\,a\,\alpha}}{(\alpha^2+\mu^2 s^2)} \tag{4.31}$$

and

$$\overline{\sigma}^{*\prime}{}_-(\alpha, 0, s) = -\, C(\alpha)\, \sqrt{\alpha^2 + \mu^2 s^2}. \tag{4.32}$$

Then, Eq. (4.31) and Eq. (4.32) give:

$$\overline{\sigma}^{*}{}_{+}(\alpha,0,s) = -\frac{\overline{\sigma_{-}}^{*\prime}(\alpha,0,s)}{\sqrt{\alpha^2+\mu^2 s^2}} + \frac{\lambda\, s^2\, I_0\, e^{i\,a\,\alpha}}{(s+\beta)\,(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} - \frac{\lambda\, k\, I_0\, e^{i\,a\,\alpha}}{(\alpha^2+\mu^2 s^2)}. \tag{4.33}$$

This equation (4.33) which is hold in the strip $\tau_{-} < \tau < \tau_{+}$ is the Wiener-Hopf equation. However, the unknown functions $\overline{\sigma}^{*}{}_{+}$ and $\overline{\sigma_{-}}^{*\prime}$ satisfying Eq. (4.33) are analytic in the upper $(\tau_{-} < \tau)$ and lower $(\tau < \tau_{+})$ half plane respectively. The solution of this equation (4.33) is presented in the next section.

4.2.5 Solution of the Wiener-Hopf equation

The goal here, is to have the terms in Eq. (4.33) to be either analytic in the upper half plane or lower half plane. This goal is achieved by decomposing or factoring the mixed terms in that equation using the theorems in [12].

Now, let,

$$M(\alpha) = \frac{1}{\sqrt{\alpha^2+\mu^2 s^2}}\,,$$

then $M(\alpha)$ can be factorized as follows,

$$M(\alpha) = M_{+}(\alpha)\, M_{-}(\alpha).$$

However, by choosing a suitable branches for the square roots in such a way that $\frac{1}{\sqrt{\alpha+i\mu s}}$ is analytic in the upper half plane ($\tau > \tau_{-}$), with $\tau_{-} > Im(-i\mu s)$. Similarly, $\frac{1}{\sqrt{\alpha-i\mu s}}$ is analytic in the lower half plane ($\tau < \tau_{+}$), with $\tau_{+} < Im(i\mu s)$. We can then deduce that:

$$\begin{cases} M_{+}(\alpha) = \frac{1}{\sqrt{\alpha+i\mu s}} \;\;, \; analytic \; in \; (\,\tau > \tau_{-}) \\ \\ M_{-}(\alpha) = \frac{1}{\sqrt{\alpha-i\mu s}} \;\;. \; analytic \; in \; (\,\tau < \tau_{+}\,) \end{cases}$$

In the same way we put,

$$N(\alpha) = \frac{\lambda\, s^2\, I_0\, e^{i\, a\, \alpha}}{(s+\beta)\,(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} - \frac{\lambda\, k\, I_0\, e^{i\, a\, \alpha}}{(\alpha^2+\mu^2 s^2)}\,.$$

Then Eq. (4.32) becomes :

$$\overline{\sigma}^{*}{}_{+}(\alpha,0,s) = -\overline{\sigma_{-}}^{*\prime}(\alpha,0,s) M_{+}(\alpha)\, M_{-}(\alpha) + N(\alpha), \tag{4.34}$$

Divide Eq. (4.33) by $M_{+}(\alpha)$ to get,

$$\frac{\overline{\sigma}^{*}{}_{+}(\alpha,0,s)}{M_{+}(\alpha)} = -\overline{\sigma_{-}}^{*\prime}(\alpha,0,s)\, M_{-}(\alpha) + \frac{N(\alpha)}{M_{+}(\alpha)}. \tag{4.35}$$

In Eq. (4.35) we only $\frac{N(\alpha)}{M_{+}(\alpha)}$ as a mixed term. We decompose this mixed term using the decomposition theorem (see Nobel [12] , page 13) to get:

$$P(\alpha) = \frac{N(\alpha)}{M_{+}(\alpha)} = P_{+}(\alpha) + \; P_{-}(\alpha),$$

where $P_{+}(\alpha)$ and $P_{-}(\alpha)$ are given to be :

$$P_{+}(\alpha) = \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\left(\frac{N(z)}{M_{+}(z)}\right)\frac{1}{z-\alpha}\,dz \tag{4.36}$$

and

$$P_{+}(\alpha) = \frac{-1}{2\pi i}\int_{-\infty+id}^{\infty+id}\left(\frac{N(z)}{M_{+}(z)}\right)\frac{1}{z-\alpha}\,dz, \tag{4.37}$$

respectively, where c and d are chosen to be within the strip or analyticity. That is, $\tau_{-} < c < Im(\alpha) = \tau \; < d < \tau_{+}$.

Finally, Eq. (4.35) becomes :

$$\frac{\overline{\sigma}^*{}_+(\alpha,0,s)}{M_+(\alpha)} - P_+(\alpha) = -\overline{\sigma_-}^{*\prime}(\alpha,0,s)\, M_-(\alpha) + P_-(\alpha). \tag{4.38}$$

We now define $E(\alpha)$ as,

$$E(\alpha) = \frac{\overline{\sigma}^*{}_+(\alpha,0,s)}{M_+(\alpha)} - P_+(\alpha) = -\overline{\sigma_-}^{*\prime}(\alpha,0,s)\, M_-(\alpha) + P_-(\alpha). \tag{4.39}$$

Then, Eq. (4.39) defines $E(\alpha)$ only in the strip $_- < \tau < \tau_+$. But the second part of the equation is defined and analytic in $> \tau_-$, and the third part is defined and analytic in $\tau < \tau_+$. Hence by analytic continuation we can define $E(\alpha)$ over the whole α-plane and write :

$$E(\alpha) = \begin{cases} \frac{\overline{\sigma}^*{}_+(\alpha,0,s)}{M_+(\alpha)} - P_+(\alpha)\ , & \tau > \tau_- \\ -\overline{\sigma_-}^{*\prime}(\alpha,0,s)\, M_-(\alpha) + P_-(\alpha)\ , & \tau < \tau_+ \\ \frac{\overline{\sigma}^*{}_+(\alpha,0,s)}{M_+(\alpha)} - P_+(\alpha) = -\overline{\sigma_-}^{*\prime}(\alpha,0,s)\, M_-(\alpha) + P_-(\alpha). & \tau_- < \tau < \tau_+ \end{cases} \tag{4.40}$$

Now, Eq. (4.40) defines an entire function $E(\alpha)$ in the whole plane. Moreover, it can be shown from the asymptotic behavior of $E(\alpha)$ which vanishes when $\| \alpha \| \to \infty$, that is $E(\alpha)$ is bounded. Hence, we deduce that by the extended form of Liouville's theorem that $E(\alpha)$ is zero.

Thus, from Eq. (4.36) we get,

$\frac{\overline{\sigma}^*{}_+(\alpha,0,s)}{M_+(\alpha)} - P_+(\alpha) = 0$, which gives,

$$\overline{\sigma}^*{}_+(\alpha,0,s) = M_+(\alpha)\, P_+(\alpha). \tag{4.41}$$

Eq. (4.31) and (4.32) give :

$C(\alpha) = \overline{\sigma}^*{}_+(\alpha,0,s) - \frac{\lambda\, s^2\, I_0\, e^{i\,a\,\alpha}}{(s+\beta)\,(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} + \frac{\lambda\, k\, I_0\, e^{i\,a\,\alpha}}{(\alpha^2+\mu^2 s^2)}$ and thus,

$$C(\alpha) = M_+(\alpha)\,P_+(\alpha) - \frac{\lambda\, s^2\, I_0\, e^{i\,a\,\alpha}}{(s+\beta)\,(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} + \frac{\lambda\, k\, I_0\, e^{i\,a\,\alpha}}{(\alpha^2+\mu^2 s^2)},$$

(4.42)

Therefore, the thermal stress in Eq. (4.28) is given by,

$$\overline{\sigma}^*(\alpha, y, s) = \left(\begin{array}{c} \left(\begin{array}{c} M_+(\alpha)\,P_+(\alpha) - \\ \frac{\lambda\, s^2\, I_0\, e^{i\,a\,\alpha}}{(s+\beta)\,(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} + \frac{\lambda\, k\, I_0\, e^{i\,a\,\alpha}}{(\alpha^2+\mu^2 s^2)} \end{array} \right) e^{-\sqrt{\alpha^2+\mu^2 s^2}\; y} \; + \\ \frac{\lambda\, s^2\, I_0\, e^{i\,a\,\alpha}}{(s+\beta)\,(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} - \frac{\lambda\, k\, I_0\, e^{i\,a\,\alpha}}{(\alpha^2+\mu^2 s^2)} \end{array} \right). \tag{4.43}$$

4.2.6 Decomposition of $P_+(\alpha)$

Recall,

$$\frac{K(\alpha)}{M_+(\alpha)} = \frac{\lambda\, s^2\, I_0\, e^{i\,a\,\alpha}}{(s+\beta)\sqrt{\alpha+i\mu s}\,(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} - \frac{\lambda\, k\, I_0\, e^{i\,a\,\alpha}}{\sqrt{\alpha+i\mu s}(\alpha^2+\mu^2 s^2)}.$$

From Eq. (4.36) we have,

$$P_+(\alpha) = \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic} \left(\begin{array}{c} \frac{\lambda\, s^2\, I_0\, e^{i\,a\,z}}{(s+\beta)\sqrt{z+i\mu s}\,(z^2+\mu^2 s^2)\left(z^2+\frac{s}{k}\right)} - \\ \frac{\lambda\, k\, I_0\, e^{i\,a\,\alpha}}{\sqrt{z+i\mu s}(z^2+\mu^2 s^2)} \end{array} \right) \frac{1}{z-\alpha}\, dz\,. \tag{4.44}$$

For this, let's consider the following closed contour

$\gamma = (\,-\infty + ic\,, \infty + ic\,) \cup C_R^+$, where C_R^+ is the semi-circle in the upper half plane since $a > 0$.

Now, let

$$G(z) = \left(\frac{\lambda\, s^2\, I_0\, e^{i\,a\,z}}{(s+\beta)\sqrt{z+i\mu s}\,(z^2+\mu^2 s^2)\left(z^2+\frac{s}{k}\right)} - \frac{\lambda\, k\, I_0\, e^{i\,a\,\alpha}}{\sqrt{z+i\mu s}(z^2+\mu^2 s^2)} \right) \frac{1}{z-\alpha},$$

then, G has the following simple poles which lie inside γ,

$z = \alpha$, $z = i\mu s$ and $z = i\sqrt{\frac{s}{k}}$.

Therefore,

$$\int_{\gamma} G(z) = \int_{-\infty+ic}^{\infty+ic} G(z) + \int_{C_R^+} G(z) = \sum_{k=1}^{3} Res[G(z), z_k] ,$$

The integral over C_R^+ vanishes due to Jordan Lemma.

Hence,

$P_+(\alpha) = \sum_{k=1}^{3} Res[G(z), z_k]$ which is given by,

$$P_+(\alpha) = \left(\frac{i e^{-a\sqrt{\frac{s}{k}}}\, I_0\, k^2 \sqrt{\frac{s}{k}}\lambda^2 \sqrt{i\left(\sqrt{\frac{s}{k}}+s\mu\right)}}{2\left(\sqrt{\frac{s}{k}}+i\alpha\right)(s+\beta)\left(\sqrt{\frac{s}{k}}+s\mu\right)(-1+ks\mu^2)} - \frac{e^{-as\mu}\, I_0\, \lambda \sqrt{is\mu}\left(-s-\beta+ks\lambda+ks^2\mu^2+ks\beta\mu^2\right)}{2\sqrt{2}s^2(s+\beta)\mu^2(-\alpha+is\mu)(-1+ks\mu^2)} + \frac{e^{ia\alpha}\, I_0\, \lambda\left(s^2+ks\alpha^2+s\beta+k\alpha^2\beta-ks^2\lambda\right)}{(s+k\alpha^2)(s+\beta)\sqrt{\alpha+is\mu}(\alpha^2+s^2\mu^2)} \right). \tag{4.45}$$

4.2.7 Closed form of the thermal-stress

From Eq. (4.43) we have :

$$\overline{\sigma}^*(\alpha, y, s) = \begin{pmatrix} \left(\begin{matrix} M_+(\alpha)\, P_+(\alpha) - \\ \frac{\lambda\, s^2\, I_0\, e^{i\,a\,\alpha}}{(s+\beta)\,(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} + \frac{\lambda\, k\, I_0\, e^{i\,a\,\alpha}}{(\alpha^2+\mu^2 s^2)} \end{matrix} \right) e^{-\sqrt{\alpha^2+\mu^2 s^2}\; y} \; + \\ \frac{\lambda\, s^2\, I_0\, e^{i\,a\,\alpha}}{(s+\beta)\,(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} - \frac{\lambda\, k\, I_0\, e^{i\,a\,\alpha}}{(\alpha^2+\mu^2 s^2)} \end{pmatrix}.$$

This solution can be to determine the overall stress effect of the body in the transformed domain. The inverse Laplace transform and the inversion Fourier transform can then be used to obtain the thermal $\sigma(x, y, t)$ at given (x, y, t).

Therefore,

$\sigma(x, y, t) =$

$$\frac{1}{4\pi^2 i}\int_{-i\infty+c}^{i\infty+c}\int_{-\infty}^{\infty} \begin{pmatrix} \left(\begin{matrix} M_+(\alpha)\, P_+(\alpha) - \\ \frac{\lambda\, s^2\, I_0\, e^{i\,a\,\alpha}}{(s+\beta)\,(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} + \frac{\lambda\, k\, I_0\, e^{i\,a\,\alpha}}{(\alpha^2+\mu^2 s^2)} \end{matrix} \right) e^{-\sqrt{\alpha^2+\mu^2 s^2}\; y} \\ + \frac{\lambda\, s^2\, I_0\, e^{i\,a\,\alpha}}{(s+\beta)\,(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} - \frac{\lambda\, k\, I_0\, e^{i\,a\,\alpha}}{(\alpha^2+\mu^2 s^2)} \end{pmatrix} e^{-i\alpha x} e^{st} d\alpha\, ds. \quad (4.46)$$

These integrals in Eq. (4.46) is the closed form solution of the thermal stress. Evaluation of these integrals analytically is not easy task. This is due to the multiple valued functions resulting from the square roots in the integrand. Moreover, the singularity at infinity is not isolated, and hence we can’t use the residue at infinity to evaluate such integrals. However, the contribution of the poles is investigated in the next section.

4.2.8 Thermal-stress due to the poles contributions

The integrals in Eq. (4.46) can be treated individually as follows:

Let

$$G(\alpha) = \left(\begin{array}{c} \left(\begin{array}{c} M_+(\alpha)\, P_+(\alpha) - \\ \frac{\lambda\, s^2\, I_0\, e^{i\,a\,\alpha}}{(s+\beta)\,(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} + \frac{\lambda\, k\, I_0\, e^{i\,a\,\alpha}}{(\alpha^2+\mu^2 s^2)} \end{array} \right) e^{-\sqrt{\alpha^2+\mu^2 s^2}\; y} \;+ \\ \frac{\lambda\, s^2\, I_0\, e^{i\,a\,\alpha}}{(s+\beta)\,(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} - \frac{\lambda\, k\, I_0\, e^{i\,a\,\alpha}}{(\alpha^2+\mu^2 s^2)} \end{array} \right), \text{ then}$$

$G(\alpha)$ has two simple poles at :

$\alpha = i\sqrt{\frac{s}{k}}$ and $\alpha = -\,i\sqrt{\frac{s}{k}}$ then we have,

$$\overline{\sigma}(x,y,s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\alpha)\, e^{-i\alpha x} = \begin{cases} Res\left[G(\alpha), -\,i\sqrt{\frac{s}{k}}\right], & x > 0 \\ Res\left[G(\alpha), i\sqrt{\frac{s}{k}}\right]. & x < 0 \end{cases}$$

Finding, we get :

$$\overline{\sigma}(x,y,s) = \frac{\left(e^{a\sqrt{\frac{s}{k}} - \sqrt{\frac{s}{k}}\,x - y\sqrt{\frac{s(-1+ks\mu^2)}{k}}}\, I_0\, s^2 \lambda^2 \left(\begin{array}{c} 1 + i\sqrt{\frac{s}{k}} - i e^{y\sqrt{\frac{s(-1+ks\mu^2)}{k}}} \sqrt{\frac{s}{k}} + \\ k\sqrt{\frac{s}{k}}\mu - iks\sqrt{\frac{s}{k}}\mu^2 + i e^{y\sqrt{\frac{s(-1+ks\mu^2)}{k}}} ks\sqrt{\frac{s}{k}}\mu^2 \end{array} \right) \right)}{2(s+\beta)\left(-\sqrt{\frac{s}{k}}+s\mu\right)^2 \left(\sqrt{\frac{s}{k}}+s\mu\right)^2}, \text{ if } x > 0 \tag{4.47}$$

and

$$\overline{\sigma}(x,y,s)=\frac{\left(\begin{array}{c} ie^{-a\sqrt{\frac{s}{k}}+\sqrt{\frac{s}{k}}x-y\sqrt{\frac{s(-1+ks\mu^2)}{k}}}\times \\ \left(-1+e^{y\sqrt{\frac{s(-1+ks\mu^2)}{k}}}\right)I_0\, s^2\sqrt{\frac{s}{k}}\lambda^2\left(\begin{array}{c}\sqrt{\frac{s}{k}}+3s\mu+\\ 3ks\sqrt{\frac{s}{k}}\mu^2+ks^2\mu^3\end{array}\right)\end{array}\right)}{2(s+\beta)\left(\sqrt{\frac{s}{k}}-s\mu\right)\left(\sqrt{\frac{s}{k}}+s\mu\right)^4}. \qquad \text{if } x<0 \tag{4.48}$$

Eq. (4.47) and (4.48) are in the Laplace domain, and the only pole that these equations have in s domain is at $s=-\beta$. Therefore, we get :

$$\sigma(x,y,t)=\frac{\left(e^{-t\beta+a\sqrt{-\frac{\beta}{k}}-x\sqrt{-\frac{\beta}{k}}-y\sqrt{\frac{\beta(1+k\beta\mu^2)}{k}}}I_0\,\beta^2\lambda^2\left(\begin{array}{c}1+i\sqrt{-\frac{\beta}{k}}-ie^{y\sqrt{\frac{\beta(1+k\beta\mu^2)}{k}}}\sqrt{-\frac{\beta}{k}}+\\ k\sqrt{-\frac{\beta}{k}}\mu+ik\beta\sqrt{-\frac{\beta}{k}}\mu^2-\\ ie^{y\sqrt{\frac{\beta(1+k\beta\mu^2)}{k}}}k\beta\sqrt{-\frac{\beta}{k}}\mu^2\end{array}\right)\right)}{2\left(-\sqrt{-\frac{\beta}{k}}+\beta\mu\right)^2\left(\sqrt{-\frac{\beta}{k}}+\beta\mu\right)^2} \quad \text{if } x>0 \tag{4.49}$$

and

$$\sigma(x,y,t)=-\frac{\left(\begin{array}{c} ie^{-t\beta-a\sqrt{-\frac{\beta}{k}}+x\sqrt{-\frac{\beta}{k}}-y\sqrt{\frac{\beta(1+k\beta\mu^2)}{k}}}\times \\ \left(-1+e^{y\sqrt{\frac{\beta(1+k\beta\mu^2)}{k}}}\right)i\beta^2\sqrt{-\frac{\beta}{k}}\lambda^2\left(\begin{array}{c}-\sqrt{-\frac{\beta}{k}}+3\beta\mu+\\ 3k\beta\sqrt{-\frac{\beta}{k}}\mu^2-k\beta^2\mu^3\end{array}\right)\end{array}\right)}{2\left(\sqrt{-\frac{\beta}{k}}-\beta\mu\right)^4\left(\sqrt{-\frac{\beta}{k}}+\beta\mu\right)} \text{ if } x<0. \tag{4.50}$$

4.3 Thermal stress in a half space with mixed boundary conditions due to an instantaneous line heat source.

4.3.1 Formulation of the problem

We consider the same equation presented in the first part, two dimensional heat conduction equation in the presence of a line type of heat source:

$$\phi_{xx}(x,y,t)+\phi_{yy}(x,y,t)=\frac{1}{k}\ \phi_t(x,y,t)+f(x,t), \tag{4.51}$$

where,

$f(x,t)=I_0\ \delta(x-a)\ \delta(t-b)$

is the heat source ,

$\delta(x-a)$ is the Dirac delta function. As for a , b are the source parameters and I_0 is a positive coefficient.

The initial condition are given by:

$$\phi(x,y,0)=0\ \ ,\ \begin{cases}-\infty<x<\ \infty\\ \ 0\le y<\ \infty\end{cases} \tag{4.52}$$

while the boundary conditions are:

$$\lim_{y\to\infty}\phi(x,y,t)=0,\ \ \begin{cases}-\infty<x<\ \infty\\ \qquad t\ge 0\end{cases}$$

$$\phi_y(x,0,t)=0\qquad .\quad \begin{cases}-\infty<x<\ \infty\\ \qquad t\ge 0\end{cases} \tag{4.53}$$

The stress coupled equation are given by,

$$\sigma_{xx}(x,y,t)+\sigma_{yy}(x,y,t)-\mu^2\sigma_{tt}(x,y,t)=\lambda\ \phi_{tt}(x,y,t), \tag{4.54}$$

where,

μ and λ are positive coefficients.

ϕ_{tt} is the second derivative resulting from the solution of Eq. (4.51).

The initial conditions are assumed as follows:

$$\sigma(x, y, 0) = 0 \ , \qquad \begin{cases} -\infty < x < \infty \\ 0 \le y < \infty \end{cases}$$

$$\sigma_t(x, y, 0) = 0 \ , \qquad \begin{cases} -\infty < x < \infty \\ 0 \le y < \infty \end{cases} \tag{4.55}$$

while the boundary conditions are:

$$\sigma(x, 0, t) = 0 \qquad , \quad \begin{cases} -\infty < x < 0 \\ t \ge 0 \end{cases}$$

$$\sigma_y(x, 0, t) = 0 \qquad , \quad \begin{cases} 0 < x < \infty \\ t \ge 0 \end{cases} \tag{4.56}$$

Also,

$$\lim_{y \to \infty} \sigma(x, y, t) = 0 \ . \qquad \begin{cases} -\infty < x < \infty \\ t \ge 0 \end{cases} \tag{4.57}$$

4.3.2 The Wiener-Hopf Equation

The Laplace transform in the time variable t and its inverse transform in s are defined by:

$$L\{\sigma(x, y, t)\} = \int_0^\infty \sigma(x, y, t) e^{-st} dt = \overline{\sigma}(x, y, s) \tag{4.58}$$

and

$$L^{-1}\{\overline{\sigma}(x, y, s)\} = \frac{1}{2\pi i} \int_{i\infty - c}^{i\infty + c} \overline{\sigma}(x, y, s) \ e^{st} ds = \sigma(x, y, t) \tag{4.59}$$

As in chapter 1, we define the Fourier transform in x and its corresponding inverse Fourier transform in α by:

$$\mathcal{F}\{\sigma(x, y, t)\} = \int_{-\infty}^{\infty} \sigma(x, y, t) e^{i\alpha x} dx = \sigma^*(\alpha, y, t) \tag{4.60}$$

and

$$\mathcal{F}^{-1}\{\sigma^*(\alpha, y, t)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sigma^*(\alpha, y, t) e^{-i\alpha x} d\alpha = \sigma(x, y, t), \tag{4.61}$$

with $\alpha = \xi + i\tau$.

Moreover, we also introduce the half range Fourier transforms as

$$\int_0^\infty \sigma(x, y, t)\, e^{i\alpha x} dx = \sigma_+^*(\alpha, y, t) \tag{4.62}$$

and

$$\int_{-\infty}^0 \sigma(x, y, t) e^{i\alpha x} dx = \sigma_-^*(\alpha, y, t), \tag{4.63}$$

So that,

$$\sigma^*(\alpha, y, t) = \sigma_+^*(\alpha, y, t) + \sigma_-^*(\alpha, y, t), \tag{4.64}$$

where,

$\sigma_+^*(\alpha, y, t) = O(e^{\tau_- x})$ as $x \to \infty$ and $T\_(z) = O(e^{\tau_+ x})$ as $x \to -\infty$, See [12].

Thus $\sigma_+^*(\alpha, y, t)$ is an analytic function of α in the upper half-plane $\tau > \tau_-$, while $\sigma_-^*(\alpha, y, t)$ is an analytic function of α in the lower half-plane $\tau < \tau_+$ respectively. Therefore, $\sigma^*(\alpha, y, t)$ defined an analytic function in the common strip $\tau_- < \tau < \tau_+$ with

$\tau = Im(\alpha)$.

4.3.3 Solution of the heat equation

Taking Laplace transform in t of Eq. (4.51) we get:

$$\bar{\phi}_{xx}(x, y, s) + \bar{\phi}_{yy}(x, y, s) = \frac{s}{k}\bar{\phi}(x, y, t) - \frac{1}{k}\phi(x, y, 0) + \bar{f}(x, s). \tag{4.65}$$

Applying the initial condition to get:

$$\bar{\phi}_{xx}(x, y, s) + \bar{\phi}_{yy}(x, y, s) = \frac{s}{k}\bar{\phi}(x, y, t) + \bar{f}(x, s). \tag{4.66}$$

Taking Fourier transform in x of Eq. (4.66) we obtain :

$$-\alpha^2\bar{\phi}^*(\alpha, y, s) + \bar{\phi}^*{}_{yy}(\alpha, y, s) = \frac{s}{k}\bar{\phi}^*(\alpha, y, t) + \bar{f}^*(\alpha, s), \tag{4.67}$$

which can be written as,

$$\bar{\phi}^{*}{}_{yy}(\alpha, y, s) - \left(\alpha^2 + \frac{s}{k}\right) \bar{\phi}^{*}(\alpha, y, s) = \bar{f}^{*}(\alpha, s). \tag{4.68}$$

Therefore the solution is

$$\bar{\phi}^{*}(\alpha, y, s) = A\, e^{-\sqrt{\alpha^2+\frac{s}{k}}\, y} + B\, e^{\sqrt{\alpha^2+\frac{s}{k}}\, y} - \frac{\bar{f}^{*}(\alpha,s)}{\left(\alpha^2+\frac{s}{k}\right)}, \tag{4.69}$$

where,

$$\bar{f}^{*}(\alpha, s) = I_0\, e^{i\, a\, \alpha}\, e^{-\, b\, s}\,.$$

Now, using boundary conditions in Eq. (4.53) we deduce that :

$B = 0$ and $A = 0$

Thus, Eq. (4.69) becomes :

$$\bar{\phi}^{*}(\alpha, y, s) = -\frac{I_0\, e^{i\, a\, \alpha}\, e^{-\, b\, s}}{\left(\alpha^2+\frac{s}{k}\right)}. \tag{4.70}$$

By Partial fraction decomposition and the second shifting theorem (SST), the inverse Laplace transform of Eq. (4.70) is found to be:

$$\phi^{*}(\alpha, y, t) = \begin{cases} -\, e^{ia\alpha - k(-b+t)\,\alpha^2} I_0\, k\,, & t > b \\ \\ 0\;. & t < b \end{cases} \tag{4.71}$$

where $u[t]$ is the unit step function.

So that $\phi(x, y, t)$ is given by:

$$\phi(x, y, t) = \begin{cases} \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi^{*}(\alpha, y, t)\, e^{-i\,\alpha\, x}\, d\alpha\,, & t > b \\ \\ 0\,. & t < b \end{cases} \tag{4.72}$$

4.3.4 Solution of the stress equation

Taking Fourier transform in x of Eq. (4.54) we obtain :

$$-\alpha^2\sigma^*(\alpha,y,t)+\sigma^*{}_{yy}(\alpha,y,t)-\mu^2\sigma^*{}_{tt}(\alpha,y,t)=\lambda\,\phi^*{}_{tt}(\alpha,y,t). \tag{4.73}$$

Now, taking Laplace transform in t we get,

$$-\alpha^2\overline{\sigma}^*(\alpha,y,s)+\overline{\sigma}^*{}_{yy}(\alpha,y,s)-\mu^2s^2\overline{\sigma}^*(\alpha,y,s)-\mu^2s\,\sigma^*(\alpha,y,0)-\mu^2\sigma^*{}_t(\alpha,y,0)=$$
$$\lambda\,s^2\bar{\phi}^*(\alpha,y,s)-\lambda\,\bar{\phi}(\alpha,y,0)-\lambda\,\overline{\phi_t}(\alpha,y,0). \tag{4.74}$$

Applying the transformed initial conditions we get :

$$-\alpha^2\overline{\sigma}^*(\alpha,y,s)+\overline{\sigma}^*{}_{yy}(\alpha,y,s)-\mu^2s^2\overline{\sigma}^*(\alpha,y,s)=\lambda\,s^2\bar{\phi}^*(\alpha,y,t)-\lambda\,\overline{\phi_t}(\alpha,y,0). \tag{4.75}$$

$\overline{\phi_t}(\alpha,y,0)$ is computed using Eq. (4.71) to get :

$$\overline{\phi_t}(\alpha,y,0)=e^{i\,a\,\alpha}e^{b\,k\,\alpha^2}I_0\,k^2\,\alpha^2\,,\quad t>b.$$

so that Eq. (4.75) becomes :

$$\overline{\sigma}^*{}_{yy}(\alpha,y,t)-(\alpha^2+\mu^2s^2)\,\overline{\sigma}^*(\alpha,y,t)=-\frac{\lambda\,s^2\,I_0\,e^{i\,a\,\alpha}\,e^{-b\,s}}{\left(\alpha^2+\frac{s}{k}\right)}+\lambda\,e^{i\,a\,\alpha}e^{b\,k\,\alpha^2}I_0\,k^2\,\alpha^2\,, \tag{4.76}$$

Therefore the solution is,

$$\overline{\sigma}^*(\alpha,y,s)=\begin{pmatrix}C(\alpha)e^{-\sqrt{\alpha^2+\mu^2s^2}\;y}+D(\alpha)\,e^{\sqrt{\alpha^2+\mu^2s^2}\;y}+\\ \frac{\lambda\,s^2\,I_0\,e^{i\,a\,\alpha}\,e^{-b\,s}}{(\alpha^2+\mu^2s^2)\left(\alpha^2+\frac{s}{k}\right)}-\frac{\lambda\,e^{i\,a\,\alpha}e^{b\,k\,\alpha^2}I_0\,k^2\,\alpha^2}{(\alpha^2+\mu^2s^2)}\end{pmatrix}. \tag{4.77}$$

Using the boundary in Eq. (4.57) we deduce that $D(\alpha)=0$.

hence,

$$\overline{\sigma}^*(\alpha,y,s)=C(\alpha)\,e^{-\sqrt{\alpha^2+\mu^2s^2}\;y}+\frac{\lambda\,s^2\,I_0\,e^{i\,a\,\alpha}\,e^{-b\,s}}{(\alpha^2+\mu^2s^2)\left(\alpha^2+\frac{s}{k}\right)}-\frac{\lambda\,e^{i\,a\,\alpha}e^{b\,k\,\alpha^2}I_0\,k^2\,\alpha^2}{(\alpha^2+\mu^2s^2)}. \tag{4.78}$$

Now, we transform the boundaries in Eq. (4.56) as follows,

$$\overline{\sigma}^{*}{}_{-}(\alpha, 0, s) = 0 \tag{4.79}$$

and

$$\overline{\sigma_{-}}^{*\prime}(\alpha, 0, s) = 0 \tag{4.80}$$

where $\{\,'\,\}$ denotes the derivative of $\overline{\sigma}^{*}$ with respect to y.

so that,

$$\overline{\sigma}^{*}{}_{+}(\alpha, 0, s) = C(\alpha) + \frac{\lambda\, s^2\, I_0\, e^{i\,a\,\alpha}\, e^{-\,b\,s}}{(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} - \frac{\lambda\, e^{i\,a\,\alpha} e^{b\,k\,\alpha^2} I_0\, k^2\, \alpha^2}{(\alpha^2+\mu^2 s^2)}\,, \tag{4.81}$$

$$\overline{\sigma}^{*\prime}{}_{-}(\alpha, 0, s) = -\,C(\alpha)\sqrt{\alpha^2+\mu^2 s^2}\,, \tag{4.82}$$

Then, Eq. (4.81) and Eq. (4.82) give:

$$\overline{\sigma}^{*}{}_{+}(\alpha, 0, s) = -\frac{\overline{\sigma_{-}}^{*\prime}(\alpha,0,s)}{\sqrt{\alpha^2+\mu^2 s^2}} + \frac{\lambda\, s^2\, I_0\, e^{i\,a\,\alpha}\, e^{-\,b\,s}}{(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} - \frac{\lambda\, e^{i\,a\,\alpha} e^{b\,k\,\alpha^2} I_0\, k^2\, \alpha^2}{(\alpha^2+\mu^2 s^2)}\,. \tag{4.83}$$

This equation (4.83) which holds in the strip $\tau_{-} < \tau < \tau_{+}$ is the Wiener-Hopf equation. However, the unknown functions $\overline{\sigma}^{*}{}_{+}$ and $\overline{\sigma_{-}}^{*\prime}$ satisfying Eq. (4.83) are analytic in the upper $(\tau_{-} < \tau)$ and lower $(\tau < \tau_{+})$ half plane respectively. The solution of this equation (4.83) is presented in the next section.

4.3.5 Solution of the Wiener-Hopf equation

The goal here, is to have the terms in Eq. (4.83) to be either analytic in the upper half plane or lower half plane. This goal is achieved by decomposing or factoring the mixed terms in that equation using the theorems in [12].

Now, as in the first part

$$M(\alpha) = M_+(\alpha)\, M_-(\alpha)$$

Same argument, as in part 1, applied to $M(\alpha)$ to get :

$$\begin{cases} M_+(\alpha) = \frac{1}{\sqrt{\alpha + i\mu s}} \ , \ analytic\ in\ (\ \tau > \tau_-) \\ \\ M_-(\alpha) = \frac{1}{\sqrt{\alpha - i\mu s}} \ , \ analytic\ in\ (\ \tau < \tau_+\ ) \end{cases}$$

Also, we let :

$$K(\alpha) = \frac{\lambda\, s^2\, I_0\, e^{i\,a\,\alpha}\, e^{-\,b\,s}}{(\alpha^2 + \mu^2 s^2)\,\left(\alpha^2 + \frac{s}{k}\right)} - \frac{\lambda\, e^{i\,a\,\alpha} e^{b\,k\,\alpha^2} I_0\, k^2\, \alpha^2}{(\alpha^2 + \mu^2 s^2)}$$

Then Eq. (4.82) becomes :

$$\overline{\sigma}^*{}_+(\alpha, 0, s) = -\overline{\sigma_-}^{*\prime}(\alpha, 0, s) M_+(\alpha)\, M_-(\alpha) + K(\alpha)\ . \qquad (4.84)$$

Dividing Eq. (4.83) by $M_+(\alpha)$ to get

$$\frac{\overline{\sigma}^*{}_+(\alpha,0,s)}{M_+(\alpha)} = -\overline{\sigma_-}^{*\prime}(\alpha, 0, s)\, M_-(\alpha) + \frac{K(\alpha)}{M_+(\alpha)}\ . \qquad (4.85)$$

In Eq. (4.85) we only $\frac{K(\alpha)}{M_+(\alpha)}$ as a mixed term. We decompose this mixed term using the decomposition theorem (Nobel [12] , page 13) to get:

$$P(\alpha) = \frac{K(\alpha)}{M_+(\alpha)} = P_+(\alpha) + P_-(\alpha)\ ,$$

where, $P_+(\alpha)$ and $P_-(\alpha)$ are given to be :

$$P_+(\alpha) = \frac{1}{2\pi i}\int_{-\infty + ic}^{\infty + ic} \left(\frac{K(z)}{M_+(z)}\right) \frac{1}{z - \alpha}\, dz \qquad (4.86)$$

and

$$P_+(\alpha) = \frac{-1}{2\pi i}\int_{-\infty+id}^{\infty+id}\left(\frac{K(z)}{M_+(z)}\right)\frac{1}{z-\alpha}dz, \qquad (4.87)$$

respectively, where c and d are chosen to be within the strip or analyticity. That is, $\tau_- < c < Im(\alpha) = \tau < d < \tau_+$.

Finally, Eq. (4.85) becomes :

$$\frac{\overline{\sigma}^*{}_+(\alpha,0,s)}{M_+(\alpha)} - P_+(\alpha) = -\overline{\sigma_-}^{*\prime}(\alpha,0,s)\,M_-(\alpha) + P_-(\alpha)\,. \qquad (4.88)$$

We now define $E(\alpha)$ as :

$$E(\alpha) = \frac{\overline{\sigma}^*{}_+(\alpha,0,s)}{M_+(\alpha)} - P_+(\alpha) = -\overline{\sigma_-}^{*\prime}(\alpha,0,s)\,M_-(\alpha) + P_-(\alpha)\,. \qquad (4.89)$$

Then, Eq. (4.89) defines $E(\alpha)$ only in the strip $_- < \tau < \tau_+$. But the second part of the equation is defined and analytic in $> \tau_-$, and the third part is defined and analytic in $\tau < \tau_+$. Hence by analytic continuation we can define $E(\alpha)$ over the whole α-plane and write :

$$E(\alpha) = \begin{cases} \frac{\overline{\sigma}^*{}_+(\alpha,0,s)}{M_+(\alpha)} - P_+(\alpha)\ , & \tau > \tau_- \\ -\overline{\sigma_-}^{*\prime}(\alpha,0,s)\,M_-(\alpha) + P_-(\alpha)\ , & \tau < \tau_+ \\ \frac{\overline{\sigma}^*{}_+(\alpha,0,s)}{M_+(\alpha)} - P_+(\alpha) = -\overline{\sigma_-}^{*\prime}(\alpha,0,s)\,M_-(\alpha) + P_-(\alpha), & \tau_- < \tau < \tau_+ \end{cases} \qquad (4.90)$$

Now, Eq. (4.90) defines an entire function $E(\alpha)$ in the whole plane. Moreover, it can be shown from the asymptotic behavior of $E(\alpha)$ which vanishes when $\| \alpha \| \to \infty$, that is $E(\alpha)$ is bounded. Hence, we deduce that by the extended form of Liouville's theorem that $E(\alpha)$ is zero.

Thus, from Eq. (4.89) we get :

$\frac{\overline{\sigma}^*{}_+(\alpha,0,s)}{M_+(\alpha)} - P_+(\alpha) = 0$ which gives:

$$\overline{\sigma}^*{}_+(\alpha,0,s) = M_+(\alpha)\,P_+(\alpha) \qquad (4.91)$$

Eq. (4.81) and (4.82) give,

$$C(\alpha) = \overline{\sigma}^{*}{}_{+}(\alpha, 0, s) - \frac{\lambda\, s^2\, I_0\, e^{i\,a\,\alpha}\, e^{-\,b\,s}}{(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} + \frac{\lambda\, e^{i\,a\,\alpha} e^{b\,k\,\alpha^2} I_0\, k^2\, \alpha^2}{(\alpha^2+\mu^2 s^2)}$$ and thus,

$$C(\alpha) = M_{+}(\alpha)\, P_{+}(\alpha) - \frac{\lambda\, s^2\, I_0\, e^{i\,a\,\alpha}\, e^{-\,b\,s}}{(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} + \frac{\lambda\, e^{i\,a\,\alpha} e^{b\,k\,\alpha^2} I_0\, k^2\, \alpha^2}{(\alpha^2+\mu^2 s^2)}, \tag{4.92}$$

Therefore, the thermal stress in Eq. (78) is given by:

$$\overline{\sigma}^{*}(\alpha, y, s) = \left(\begin{array}{c} e^{-\sqrt{\alpha^2+\mu^2 s^2}\; y} \left(\begin{array}{c} M_{+}(\alpha)\, P_{+}(\alpha) - \\ \frac{\lambda\, s^2\, I_0\, e^{i\,a\,\alpha}\, e^{-\,b\,s}}{(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} + \frac{\lambda\, e^{i\,a\,\alpha} e^{b\,k\,\alpha^2} I_0\, k^2\, \alpha^2}{(\alpha^2+\mu^2 s^2)} \end{array} \right) + \\ \frac{\lambda\, s^2\, I_0\, e^{i\,a\,\alpha}\, e^{-\,b\,s}}{(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} - \frac{\lambda\, e^{i\,a\,\alpha} e^{b\,k\,\alpha^2} I_0\, k^2\, \alpha^2}{(\alpha^2+\mu^2 s^2)} \end{array} \right). \tag{4.93}$$

4.3.6 Decomposition of $P_{+}(\alpha)$

Recall that,

$$\frac{K(\alpha)}{M_{+}(\alpha)} = \frac{\lambda\, s^2\, I_0\, e^{i\,a\,\alpha}\, e^{-\,b\,s}}{\sqrt{\alpha+i\mu s}\,(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} - \frac{\lambda\, e^{i\,a\,\alpha} e^{b\,k\,\alpha^2} I_0\, k^2\, \alpha^2}{\sqrt{\alpha+i\mu s}(\alpha^2+\mu^2 s^2)}$$

and so, from Eq. (4.86) we have

$$P_{+}(\alpha) = \frac{1}{2\pi i} \int_{-\infty+ic}^{\infty+ic} \left(\begin{array}{c} \frac{\lambda\, s^2\, I_0\, e^{i\,a\,\alpha}\, e^{-\,b\,s}}{\sqrt{\alpha+i\mu s}\,(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} - \\ \frac{\lambda\, e^{i\,a\,\alpha} e^{b\,k\,\alpha^2} I_0\, k^2\, \alpha^2}{\sqrt{\alpha+i\mu s}(\alpha^2+\mu^2 s^2)} \end{array} \right) \frac{1}{z-\alpha}\, dz. \tag{4.94}$$

For this, let's consider the following closed contour

$\gamma = (\,-\infty + ic\,, \infty + ic\,) \cup C_R^{+}$, where C_R^{+} is the semi-circle in the upper half plane since $a > 0$. Now, let

$$G(z) = \left(\frac{\lambda\, s^2\, I_0\, e^{i\,a\,\alpha}\, e^{-\,b\,s}}{\sqrt{\alpha+i\mu s}\,(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} - \frac{\lambda\, e^{i\,a\,\alpha} e^{b\,k\,\alpha^2} I_0\, k^2\, \alpha^2}{\sqrt{\alpha+i\mu s}(\alpha^2+\mu^2 s^2)} \right) \frac{1}{z-\alpha} \;,$$

then G has the following simple poles which lie inside γ.

$z = \alpha$, $z = i\mu s$ and $z = i\sqrt{\frac{s}{k}}$. Therefore,

$$\int_\gamma G(z) = \int_{-\infty+ic}^{\infty+ic} G(z) + \int_{C_R^+} G(z) = \sum_{k=1}^{3} Res[G(z), z_k].$$

The integral over C_R^+ vanishes due to Jordan Lemma.

Hence,

$P_+(\alpha) = \sum_{k=1}^{3} Res[G(z), z_k]$ which is given by:

$$P_+(\alpha) = \left(\begin{array}{c} \dfrac{i e^{-bs-a\sqrt{\frac{s}{k}}}\, I_0\, k^2 \sqrt{\frac{s}{k}}\,\lambda \sqrt{i\left(\sqrt{\frac{s}{k}} + s\mu\right)}}{2\left(\sqrt{\frac{s}{k}} + i\alpha\right)\left(\sqrt{\frac{s}{k}} + s\mu\right)(-1 + ks\mu^2)} - \\ \dfrac{e^{-bs+ia\alpha}\, I_0\, k\left(e^{bs+bk\alpha^2} ks\alpha^2 + e^{bs+bk\alpha^2} k^2\alpha^4 - s^2\lambda\right)}{(s + k\alpha^2)\sqrt{\alpha + is\mu}(\alpha^2 + s^2\mu^2)} + \\ \dfrac{e^{-bs-as\mu-bks^2\mu^2} I_0\, k\sqrt{is\mu}\left(-e^{bs}k\alpha^2 + e^{bks^2\mu^2} s\lambda + e^{bs}k^2 s\alpha^2\mu^2\right)}{2\sqrt{2}s^2\mu^2(-\alpha + is\mu)(-1 + ks\mu^2)} \end{array} \right)$$

(4.95)

4.3.7 Closed form of the thermal-stress

In Eq. (4.93), The solution can be used to determine the overall stress effect of the body in the transformed domain. The inverse Laplace transform and the inversion Fourier transform can then be used to obtain the thermal $\sigma(x, y, t)$ at given (x, y, t).

Therefore,

$$\sigma(x,y,t) = \frac{1}{4\pi^2 i}\int_{-i\infty+c}^{i\infty+c}\int_{-\infty}^{\infty}\left(\begin{array}{c} e^{-\sqrt{\alpha^2+\mu^2 s^2}\,y}\left(\begin{array}{c} M_+(\alpha)\,P_+(\alpha) - \\ \dfrac{\lambda\, s^2\, I_0\, e^{i\,a\,\alpha}\, e^{-\,b\,s}}{(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} + \\ \dfrac{\lambda\, e^{i\,a\,\alpha} e^{b\,k\,\alpha^2} I_0\, k^2\,\alpha^2}{(\alpha^2+\mu^2 s^2)} \end{array}\right) + \\ \dfrac{\lambda\, s^2\, I_0\, e^{i\,a\,\alpha}\, e^{-\,b\,s}}{(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} - \dfrac{\lambda\, e^{i\,a\,\alpha} e^{b\,k\,\alpha^2} I_0\, k^2\,\alpha^2}{(\alpha^2+\mu^2 s^2)} \end{array}\right) e^{-i\alpha x} e^{st}\, d\alpha\, ds \tag{4.96}$$

These integrals in Eq. (4.96) is the closed form solution of the thermal stress. Evaluation of these integrals analytically is not easy task. This is due to the multiple valued functions resulting from the square roots in the integrand. Moreover, the singularity at infinity is not isolated, and hence we can't use the residue at infinity to evaluate such integrals. However, the contribution of the poles is investigated in the next section.

4.3.8 Thermal-stress due to the poles contributions

The integrals in Eq. (4.96) can be treated individually as follows:

Let

$$G(\alpha) = \left(\begin{array}{c} e^{-\sqrt{\alpha^2+\mu^2 s^2}\,y}\left(\begin{array}{c} M_+(\alpha)\,P_+(\alpha) - \\ \dfrac{\lambda\, s^2\, I_0\, e^{i\,a\,\alpha}}{(s+\beta)\,(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} + \dfrac{\lambda\, e^{i\,a\,\alpha} e^{b\,k\,\alpha^2} I_0\, k^2\,\alpha^2}{(\alpha^2+\mu^2 s^2)} \end{array}\right) + \\ \dfrac{\lambda\, s^2\, I_0\, e^{i\,a\,\alpha}}{(s+\beta)\,(\alpha^2+\mu^2 s^2)\left(\alpha^2+\frac{s}{k}\right)} - \dfrac{\lambda\, e^{i\,a\,\alpha} e^{b\,k\,\alpha^2} I_0\, k^2\,\alpha^2}{(\alpha^2+\mu^2 s^2)} \end{array}\right), \text{ then}$$

$G(\alpha)$ has two simple poles at :

$\alpha = i\sqrt{\frac{s}{k}}$ and $\alpha = -\, i\sqrt{\frac{s}{k}}$ then we have

$$\overline{\sigma}(x,y,s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\alpha)\, e^{-i\alpha x} = \begin{cases} Res\left[G(\alpha), -\, i\sqrt{\frac{s}{k}}\right], & x > 0 \\ \\ Res\left[G(\alpha), i\sqrt{\frac{s}{k}}\right] \ . & x < 0 \end{cases}$$

Finding, we get :

$$\overline{\sigma}(x,y,s) = \frac{\left(e^{-bs+a\sqrt{\frac{s}{k}}-\sqrt{\frac{s}{k}}x-y\sqrt{\frac{s(-1+ks\mu^2)}{k}}} I_0 s^2 \lambda^2 \left(\begin{array}{c} 1+i\sqrt{\frac{s}{k}}-ie^{y\sqrt{\frac{s(-1+ks\mu^2)}{k}}}\sqrt{\frac{s}{k}}+k\sqrt{\frac{s}{k}}\mu - \\ iks\sqrt{\frac{s}{k}}\mu^2+ie^{y\sqrt{\frac{s(-1+ks\mu^2)}{k}}}ks\sqrt{\frac{s}{k}}\mu^2 \end{array}\right)\right)}{2\left(-\sqrt{\frac{s}{k}}+s\mu\right)^2\left(\sqrt{\frac{s}{k}}+s\mu\right)^2} \quad \text{if } \ x > 0 \qquad (4.97)$$

and

$$\overline{\sigma}(x,y,s) = \frac{\left(ie^{-bs-a\sqrt{\frac{s}{k}}+\sqrt{\frac{s}{k}}x-y\sqrt{\frac{s(-1+ks\mu^2)}{k}}}\left(-1+e^{y\sqrt{\frac{s(-1+k\,s\mu^2)}{k}}}\right) I_0\, s^2\sqrt{\frac{s}{k}}\,\lambda^2\left(\sqrt{\frac{s}{k}}+3s\mu+3ks\sqrt{\frac{s}{k}}\mu^2+ks^2\mu^3\right)\right)}{2\left(\sqrt{\frac{s}{k}}-s\mu\right)\left(\sqrt{\frac{s}{k}}+s\mu\right)^4}.$$

if $x < 0$ (4.98)

Eq. (4.97) and (4.98) are in the Laplace domain. Also they don't have isolated singularity in s and therefore the issue of the residues can't be discussed.

Chapter 5

Conclusion and Recommendations

5.1 Conclusion

The analytical solution of the hyperbolic heat conduction equation is obtained under laser short-pulse heating of solid surface situation. In chapter 2, the full pulse laser source is incorporated as a volumetric source in the equation. The general form of the analytical solution is presented using both the Fourier cosine and Laplace transform methods. Temperature variation with the time and space is computed from the analytical solution for the two parameters (β and γ) of the full pulse laser source. It is observed that as the ratio of two parameters decreases the solution tends to that solution due to the time exponentially decaying laser pulse obtained in the previous study [20]. Temperature remains high in the surface vicinity of the substrate material because of internal energy increase during absorption of the incident laser energy in this region. Since the absorption of the laser intensity inside the substrate material is governed by the Lambert's Beer law, as the depth below the surface increases temperature reduces. In this case, heat conduction from the surface region to the solid bulk becomes important to increase temperature below the surface. Since laser pulse intensity varies with time in the exponential form, increasing time alters temperature distribution inside the substrate material. In addition, lowering laser pulse parameter (γ/β) enhances temperature rise due to increased laser pulse intensity. Thermal stress is compressive inside the substrate material and remains high in the surface region. This behavior is associated with the zero stress gradient boundary at the surface, which resembles constraint for free expansion of the surface. Temporal variation of thermal stress resembles the wave behavior; in which case, thermal stress propagates in the substrate material with a constant speed ($h = \sqrt{\frac{(1+v)(1-2\,v)\rho}{E\,(1-v)}}$). Moreover, we consider the hyperbolic heat conduction equation associated with convective boundary condition under laser short-pulse heating of solid surface situation. The full pulse laser source is incorporated as a volumetric source in the equation.

In chapter 3, the step input pulse laser source has been incorporated as volumetric source in the hyperbolic heat equation. Due resulting thermal stress has been found and as in chapter 2, graphical results have been presented. It is found that the temperature distribution and the thermal stress are in accordance with nature of the source, i.e. they exhibit the step behavior.

In chapter 4, a mixed boundary value problem in thermal stress in a half space has been considered due to two different heat sources. In both cases, The heat conduction equation is solved using the Laplace transform in time and the Fourier transform in space variable. To deal with the determination of thermal stress we employ the Jones modification of the so-called Wiener-Hopf technique. The closed form solutions are obtained in the transformed domains. However, the inversion of the stress solutions seem to be tedious analytically. This is because of the type of the singularities present in the integrands due to multi-valued functions.

5.2 Recommendations

There are a number of directions which one can follow to extend the work presented here. These are outlined in the following :

1. In the hyperbolic model, the volumetric source of step input type can be studied in the case of a convective boundary condition. In this situation the temperature is related to its gradient at the surface.

2. A thermo-mechanical coupling can be incorporated in all the models.

3. The mixed boundary value problem discussed in chapter 4 can be formulated and studied for a plate of uniform thickness. Other possible extension can be to have a layered plate of two dissimilar materials.

4. The mixed boundary value problem for the hyperbolic model would be very interesting and a challenging problem for future work.

Appendix

A1. Fourier cosine transform and Laplace Transform for the derivative

$L\{u_t(x,t)\} = p\bar{u}(x,p) - u(x,0)$,

$L\{u_{tt}(x,t)\} = p^2\bar{u}(x,p) - pu(x,0) - u_t(x,0)$,

$L\{e^{-at}\} = \frac{1}{p+a} \quad a > 0$,

$F_c\{u_{xx}(x,t)\} = -s^2U(s,t) - u_x(0,t)$,

where ,

$L\{u(x,t)\} = \bar{u}(x,p)$ and

A2. Partial fraction decomposition

From Eq. (2.13)

$U^*(s,p) = \frac{I_0\delta}{(s^2+\delta^2)}(\frac{a}{P+\beta} + \frac{b}{P-r_1} + \frac{c}{P-r_2})$,

then a , b and c can be found as follows:

$a = \lim_{p\to-\beta}(p+\beta)\frac{1}{(p+\beta)(Ap^2+Bp+s^2)} = \frac{1}{A\beta^2-B\beta+s^2}$,

$b = \lim_{p\to r_1}(p-r_1)\frac{1}{(p+\beta)(Ap^2+Bp+s^2)} = \frac{1}{(r_1+\beta)(r_1-r_2)} = \frac{1}{A\beta^2-B\beta+s^2}\frac{2A\beta-B+\sqrt{B^2-4As^2}}{2\sqrt{B^2-4As^2}}$,

$c = \lim_{p\to r_2}(p-r_2)\frac{1}{(p+\beta)(Ap^2+Bp+s^2)} = \frac{1}{(r_2+\beta)(r_2-r_1)} = -\frac{1}{A\beta^2-B\beta+s^2}\frac{2A\beta-B-\sqrt{B^2-4As^2}}{2\sqrt{B^2-4As^2}}$.

A3. Investigation of the singularities type

Regarding the singularity of $U(s,t)$ due to $\sqrt{B\beta - A\beta^2}$, we inspect the Laurent series of this function at $s_1 = \sqrt{B\beta - A\beta^2}$ as follows :

$$U(s,t) = a_0 + a_1\left(s - \sqrt{B\beta - A\beta^2}\right) + a_2\left(s - \sqrt{B\beta - A\beta^2}\right)^2 + O\left[\left(s - \sqrt{B\beta - A\beta^2}\right)^3\right] .$$

Mathematica has been used to find these coefficients in the expansion above.

As we find, there is no principal part in the Laurent expansion which consists of only the analytic part. Thus we conclude that s_1 is a removable singularity. Moreover, as for $s_2 = \frac{B}{2\sqrt{A}}$ is a point beyond which $\sqrt{B^2 - 4As^2}$ becomes imaginary giving us oscillatory integral. The contribution to the integral for $s_2 > \frac{B}{2\sqrt{A}}$ vanishes due to the self-cancellation effect. We therefore restrict our range of integration to $\left[0\,,\ \frac{B}{2\sqrt{A}}\right]$. A similar argument can be applied to $V(s,t)$, $\overline{\emptyset_1}(s,t)$ and $\overline{\emptyset_2}(s,t)$ in chapter 3, to deduce the required result.

A4. The function $\mathrm{K}(s,t)$

let $\xi = \sqrt{B^2 - 4As^2}$ and $\chi = e^{\frac{\xi t}{A}}$ then

$\mathrm{K}_1(s,t) =$

$$\left[\begin{array}{c} m^2\begin{pmatrix} B(-1+\chi)+ \\ (1+\chi)\xi \end{pmatrix}\begin{pmatrix} B\,T_0 h(-s^2+B\beta)(s^2+\delta^2)\tau \\ -\ I_0 k s^2\delta(s^2+B\tau) \end{pmatrix} + \\ A^3\beta\tau\begin{pmatrix} -2(-1+\chi)\ I_0 k s^2\delta\tau - \\ T_0 h\beta(s^2+\delta^2)\begin{pmatrix} 2(-1+\chi)s^2 - \\ (B - B\chi + (1+\chi)\xi)\tau \end{pmatrix} \end{pmatrix} \end{array}\right],$$

$\mathrm{K}_2(s,t) =$

$$\left[\begin{array}{c} 2(-1+\chi)s^4 + \beta\tau\begin{pmatrix} -B^2(-1+\chi)+(1+\chi)m^2\xi\beta \\ + \\ B\big(\xi - m^2\beta + \chi(\xi + m^2\beta)\big) \end{pmatrix} - \\ s^2\begin{pmatrix} 2(-1+\chi)m^2\beta^2 + B(-1+\chi)(2\beta-\tau) + \\ (1+\chi)\xi\tau \end{pmatrix} \end{array}\right],$$

$\mathrm{K}_3(s,t) =$

$$\left[\begin{array}{c} 2(-1+\chi)s^4\beta + B(B - B\chi + (1+\chi)\xi)\beta\tau^2 - \\ s^2\tau\begin{pmatrix} \big(\xi - 2m^2\beta + \chi(\xi + 2m^2\beta)\big)\tau - \\ B(-1+\chi)(2\beta+\tau) \end{pmatrix} \end{array}\right],$$

$\mathrm{K}_4(s,t) =$

$$\left[\begin{array}{c} -B^3(-1+\chi)\ I_0\ k\beta\delta\tau + B^2\begin{pmatrix} T_0(-1+\chi)hm^2\beta(s^2+\delta^2)(\beta-\tau)\tau - \\ I_0\ k\delta\big((-1+\chi)s^2(\beta-\tau) - (1+\chi)\xi\beta\tau\big) \end{pmatrix} \\ + \\ s^2\begin{pmatrix} -\ T_0hm^2(s^2+\delta^2)\tau(2(-1+\chi)s^2 - (1+\chi)\xi\tau) - \\ I_0\ k\delta\begin{pmatrix} s^2\big(\xi - 2m^2\beta + \chi(\xi + 2m^2\beta)\big) - \\ (1+\chi)m^2\xi\tau^2 \end{pmatrix} \end{pmatrix} \end{array}\right].$$

$\mathrm{K}_5(s,t) =$

$$\left[\begin{array}{c} T_0\ hm^2(s^2+\delta^2)\tau\begin{pmatrix} (1+\chi)\xi\beta(\beta-\tau) + \\ (-1+\chi)s^2(2\beta+\tau) \end{pmatrix} + \\ I_0ks^2\delta\begin{pmatrix} (-1+\chi)s^2 + \\ \beta\big(\xi + 2m^2\tau + \chi(\xi - 2m^2\tau)\big) - \\ \tau\big(\xi + m^2\tau + \chi(\xi - m^2\tau)\big) \end{pmatrix} \end{array}\right].$$

Nomenclature

T	Temperature
σ	Thermal stress
I_0	laser peak power intensity ($W/_{m^2}$)
δ	absorption coefficient ($1/_{m}$)
h	heat transfer coefficient ($W/_{m^2K}$)
β , γ , C_1 and C_2	laser pulse parameters
ρ	density
C_p	specific heat
k	thermal conductivity
τ	relaxation time
s	Fourier transform parameter
t	time
p	Laplace transform parameter
PDE	Partial differential equation
SST	second shifting theorem

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